

Question 6

Let $T > 0$ and let g be continuous on $[0, T]$. Consider

$$y'' - y = g(t), \quad y(0) = y'(0) = 0.$$

Use the homogeneous basis e^t, e^{-t} .

Tasks

1. Derive a single definite-integral formula for the response by variation of parameters, and verify it by differentiation.
2. If $|g(t)| \leq M$ with $M > 0$, find a sharp bound for $|y(t)|$ on $[0, T]$ and exhibit forcing that attains it.
3. If $g \geq 0$, prove $y \geq 0$ and give a precise condition guaranteeing $y(t) > 0$ at a specified time $t > 0$.
4. For $g(t) = (1 + \cos 2t)/2$, find an elementary response and compare it with the bound for $M = 1$.

Question 6 – Solution

Strategy. The sign of the integral kernel gives comparison and error bounds without evaluating each forcing integral.

Step 1: Derive the response kernel. Here $W = -2$, so $u'_1 = e^{-t}g/2$ and $u'_2 = -e^tg/2$. Integrating from zero and combining gives

$$y(t) = \int_0^t \sinh(t-s)g(s) ds.$$

Differentiation gives $y' = \int_0^t \cosh(t-s)g(s) ds$ and $y'' = g(t) + y(t)$. Both initial data vanish.

Step 2: Prove sharpness. For $0 \leq s \leq t$, $\sinh(t-s) \geq 0$. Consequently

$$|y(t)| \leq M \int_0^t \sinh(t-s) ds = M(\cosh t - 1).$$

Constant forcing $g = M$ gives equality for every t , and $g = -M$ gives equality in magnitude. The bound is therefore sharp for this forcing class; it also bounds response error when g denotes a forcing error.

Step 3: Prove positivity carefully. If $g \geq 0$, the integral is nonnegative. At fixed $t > 0$, it is strictly positive exactly when g is not identically zero on $[0, t]$. Indeed, continuity then gives a positive subinterval lying before t , where the kernel is strictly positive. Forcing that is positive only at later times cannot make the response positive at this time.

Step 4: Evaluate a comparison example. For $g = (1 + \cos 2t)/2$, an elementary particular solution is $-1/2 - \cos(2t)/10$. Fitting zero data gives

$$y = \frac{3}{5} \cosh t - \frac{1}{2} - \frac{1}{10} \cos 2t.$$

Direct differentiation yields $y'' - y = (1 + \cos 2t)/2$ and zero data. Since $0 \leq g \leq 1$, the kernel proves $0 < y(t) < \cosh t - 1$ for $t > 0$: both g and $1 - g$ are nonzero on every such interval.

