

Question 4

Let g be continuous on $\mathbb{R}$. For $y'' + y = g(t)$, compare the homogeneous bases

$$(y_1, y_2) = (\cos t, \sin t), \quad (z_1, z_2) = (\cos t + \sin t, \cos t - \sin t).$$

In both parameter constructions, set the two parameter values to zero at $t = 0$.

Tasks

1. Derive the parameter derivatives for both bases, including the sign of each Wronskian.
2. Prove directly that both constructions give the same integral formula for the zero-data response.
3. If the lower endpoint is changed from zero to a , express the difference between the two particular solutions as a homogeneous combination.
4. Apply the formula to $g(t) = e^{t^2}$ with $y(0) = 2$, $y'(0) = -1$. Verify the solution without assuming that its integrals have elementary antiderivatives.

Question 4 – Solution

Strategy. A basis changes the parameter functions, but the integral response with fixed initial data must remain the same.

Step 1: Compute both parameter systems. For the first basis $W = 1$, so $u'_1 = -g \sin t$, $u'_2 = g \cos t$. The constant change-of-basis determinant is -2 , hence $W_z = -2$. Thus

$$v'_1 = \frac{1}{2}g(\cos t - \sin t), \quad v'_2 = -\frac{1}{2}g(\cos t + \sin t).$$

The negative Wronskian is essential to these signs.

Step 2: Compare the reconstructed functions. For the first basis, the integrand multiplying $g(s)$ is $\sin t \cos s - \cos t \sin s = \sin(t - s)$. For the second, it is

$$\frac{1}{2}[(\cos t + \sin t)(\cos s - \sin s) - (\cos t - \sin t)(\cos s + \sin s)],$$

which expands to the same expression. Therefore both give

$$Y_0(t) = \int_0^t \sin(t-s)g(s) ds.$$

This also holds for negative t using oriented integrals.

Step 3: Change the lower endpoint. Set $Y_a(t) = \int_a^t \sin(t-s)g(s) ds$. Then

$$Y_a - Y_0 = \cos t \int_0^a \sin s g(s) ds - \sin t \int_0^a \cos s g(s) ds.$$

Both coefficients are constants in t . Changing the lower endpoint changes a particular solution only by a homogeneous term; the data must be fitted again.

Step 4: Use a nonelementary forcing. The requested solution is

$$y = 2 \cos t - \sin t + \int_0^t \sin(t-s)e^{s^2} ds.$$

Leibniz differentiation gives $Y'_0 = \int_0^t \cos(t-s)e^{s^2} ds$ and $Y''_0 = e^{t^2} - Y_0$. Also $Y_0(0) = Y'_0(0) = 0$. Thus the equation and data hold. A definite-integral formula is a complete exact solution even without an elementary antiderivative.