

Question 3

On $\mathbb{R}$, consider

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, \quad y(0) = y'(0) = 0.$$

Use $y_1 = e^t$ and $y_2 = te^t$.

Tasks

1. Compute the Wronskian and use variation of parameters to solve the initial-value problem.
2. Independently verify the result by writing $y = e^t F(t)$ and simplifying the differential equation for F .
3. Prove that the selected response is strictly positive for every $t > 0$.
4. Find $\lim_{t \rightarrow \infty} y(t)/(te^t)$ and interpret the effect of the factor $1/(1+t^2)$ in the forcing.

Question 3 – Solution

Strategy. The repeated homogeneous root is handled by the Wronskian; an exponential substitution provides an independent residual check.

Step 1: Compute and integrate. The Wronskian is $e^t(1+t)e^t - te^{2t} = e^{2t}$. Hence

$$u_1' = -\frac{t}{1+t^2}, \quad u_2' = \frac{1}{1+t^2}.$$

Integrating from zero gives $u_1 = -\frac{1}{2}\ln(1+t^2)$ and $u_2 = \arctan t$. Therefore

$$y = e^t[t \arctan t - \frac{1}{2}\ln(1+t^2)].$$

Zero integration endpoints ensure $y(0) = y'(0) = 0$.

Step 2: Check by a different calculation. Writing $y = e^t F$ gives $y'' - 2y' + y = e^t F''$. For the displayed bracket,

$$F' = \arctan t, \quad F'' = \frac{1}{1+t^2}, \quad F(0) = F'(0) = 0.$$

These identities verify the equation and both data independently. All coefficients and the forcing are continuous on $\mathbb{R}$, so this is the unique global IVP solution.

Step 3: Prove the sign. For $t > 0$, $F(t) = \int_0^t \arctan s \, ds > 0$. Equivalently, $F(t) = \int_0^t (t-s)/(1+s^2) \, ds$ has a positive integrand in the interior. Since $e^t > 0$, the response is strictly positive.

Step 4: Compare the growth rate. For $t > 0$,

$$\frac{y}{te^t} = \arctan t - \frac{\ln(1+t^2)}{2t} \rightarrow \boxed{\pi/2}.$$

Although the forcing is smaller than e^t by a factor tending to zero, the zero-data response is asymptotic to $(\pi/2)te^t$. The graph shows the transformed response $F = e^{-t}y$, whose slope tends to $\pi/2$; it is not a plot of y itself.

