

Question 2

Solve on $I = (-\pi/2, \pi/2)$:

$$y'' + y = \sec t, \quad y(0) = y'(0) = 0.$$

Use the homogeneous pair $\cos t, \sin t$.

Tasks

1. Derive the parameter derivatives and evaluate the integrals with lower endpoint zero.
2. Simplify the initial-value solution and verify its equation and both data.
3. Determine the limits of y and y' as $t \rightarrow (\pi/2)^-$. Can the solution extend as a C^1 function through that endpoint?
4. Explain why a finite limit of y does not enlarge the interval on which this initial-value problem has a classical solution.

Question 2 – Solution

Strategy. The forcing is singular at the interval endpoints. Keep its domain visible even when the solution itself has a finite limit.

Step 1: Integrate the parameter system. Here $W = 1$. The equations $u_1' \cos t + u_2' \sin t = 0$ and $-u_1' \sin t + u_2' \cos t = \sec t$ give

$$u_1' = -\tan t, \quad u_2' = 1.$$

Since $\cos t > 0$ on I , integration from zero yields $u_1 = \ln(\cos t)$, $u_2 = t$.

Step 2: Verify the response. The zero lower endpoints fit both data, and

$$y = \cos t \ln(\cos t) + t \sin t.$$

Differentiation simplifies to

$$y' = -\sin t \ln(\cos t) + t \cos t, \quad y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t \sin t.$$

Therefore $y'' + y = (\sin^2 t + \cos^2 t)/\cos t = \sec t$, and both data vanish.

Step 3: Examine the endpoint. Using $x \ln x \rightarrow 0$ as $x \rightarrow 0^+$, we obtain

$$\lim_{t \rightarrow (\pi/2)^-} y(t) = \pi/2, \quad \lim_{t \rightarrow (\pi/2)^-} y'(t) = +\infty.$$

Thus even a C^1 extension is impossible. The graph approaches a finite height with an unbounded positive slope; the endpoint is excluded.

Step 4: Keep the equation's domain. The right side $\sec t$ is undefined at $\pi/2$ and $-\pi/2$. A continuous assignment to y at an endpoint would not define the differential equation there. The maximal open interval containing zero on which this IVP is classical is exactly I .

