

Question 1

On $t > 0$, consider

$$t^2 y'' - 2ty' + 2y = t^3, \quad y(1) = y'(1) = 0.$$

Two homogeneous solutions are $y_1 = t$ and $y_2 = t^2$.

Tasks

1. Normalize the equation and compute the Wronskian. Derive the two equations for the parameter derivatives using the auxiliary condition $u_1' y_1 + u_2' y_2 = 0$.
2. Use variation of parameters to find a particular solution and then fit the initial data.
3. Check the final solution in the original, unnormalized equation and at the initial point.
4. A student inserts t^3 , rather than the normalized forcing, into the parameter formulas. Find the resulting particular expression and its actual forcing in the original equation. Explain the error.

Question 1 – Solution

Strategy. Divide by the leading coefficient before solving for the parameter derivatives; check the answer in the original equation.

Step 1: Normalize and derive. The normalized forcing is $g(t) = t$, and $W = t(2t) - t^2 = t^2 \neq 0$. With $y = u_1t + u_2t^2$, impose $tu'_1 + t^2u'_2 = 0$. Differentiating after this cancellation gives

$$tu'_1 + t^2u'_2 = 0, \quad u'_1 + 2tu'_2 = t.$$

Thus $u'_1 = -t$ and $u'_2 = 1$.

Step 2: Integrate and fit. Choose $u_1 = -t^2/2$, $u_2 = t$; constants of integration contribute only homogeneous terms. Hence $y_p = t^3/2$ and

$$y = Ct + Dt^2 + t^3/2.$$

The data give $C + D = -1/2$ and $C + 2D = -3/2$, so $C = 1/2$, $D = -1$. Therefore

$$y = \frac{1}{2}t(t-1)^2.$$

Step 3: Verify the original equation. For the operator $L[y] = t^2y'' - 2ty' + 2y$, one has $L[t] = L[t^2] = 0$ and $L[t^3] = 2t^3$. Thus $L[y] = t^3$. Also $y(1) = 0$ and $y' = (3t^2 - 4t + 1)/2$, so $y'(1) = 0$. The normalized coefficients are continuous on $(0, \infty)$, giving uniqueness there.

Step 4: Diagnose the missing division. Using $g = t^3$ instead gives $u'_1 = -t^3$, $u'_2 = t^2$ and

$$\tilde{y}_p = -t^5/4 + t^5/3 = t^5/12.$$

Since $L[t^5] = (20 - 10 + 2)t^5 = 12t^5$, this expression satisfies $L[\tilde{y}_p] = t^5$, not t^3 . Adding homogeneous terms cannot repair its forcing. The parameter formulas apply to the normalized equation; omitting the division multiplies the intended original forcing by t^2 .