

Question 10

A prescribed acceleration changes abruptly at $t = 1$:

$$y'' = \begin{cases} 0, & 0 \leq t < 1, \\ 2, & t > 1, \end{cases} \quad y(0) = 0, \quad y'(0) = 1.$$

Seek a function that is C^1 on $[0, \infty)$ and C^2 on each open side of 1, satisfying the equation there. No value for y'' at 1 is assumed. Thus position and velocity must be continuous at the switch.

Tasks

1. Integrate on the first phase and find the position and velocity arriving at $t = 1$.
2. Integrate on the second phase with two new constants, determine them using the matching conditions, and give the complete piecewise trajectory.
3. Compute the one-sided derivatives of velocity at the switch. Can any assigned value of the forcing at $t = 1$ make this trajectory a classical C^2 solution through the switch?
4. Sketch position and velocity. A student adds $k(t - 1)$ to the second-phase position while leaving the first phase unchanged. Determine which k satisfy the stated requirements, and explain why position matching alone is insufficient.

Question 10 – Solution

Strategy. Carry both position and velocity across the switch, and distinguish a piecewise classical solution from one that is C^2 everywhere.

Step 1: Propagate the first phase. With zero acceleration and initial velocity 1, $y' = 1$ and $y = t$ for $0 \leq t \leq 1$. The arriving data are

$$\boxed{y(1) = 1, \quad y'(1) = 1}.$$

Step 2: Use both matching conditions. Writing the second-phase antiderivative about the switch gives

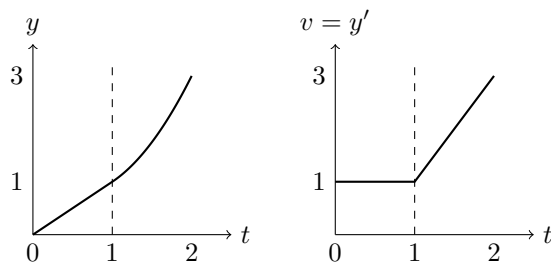
$$y = (t - 1)^2 + A(t - 1) + B \quad (t > 1).$$

Continuity of position sets $B = 1$; continuity of velocity sets $A = 1$. Thus

$$\boxed{y(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ t + (t - 1)^2, & t > 1, \end{cases}} \quad y'(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 1 + 2(t - 1), & t > 1. \end{cases}$$

These formulas satisfy all the stated requirements and uniquely determine both phases.

Step 3: Check the actual regularity. The derivative of velocity at 1 has left-hand value 0 and right-hand value 2. Therefore $y''(1)$ does not exist. Assigning a forcing value at that single point cannot change the unequal one-sided limits. The trajectory is C^1 and piecewise C^2 , but is not a classical C^2 solution across the switch.



Position has a continuous tangent; velocity has a corner.

Step 4: Reject an unmatched velocity. Adding $k(t - 1)$ on $t > 1$ keeps the position continuous and does not change the second derivative on that open phase. However, the right-hand velocity at 1 becomes $1 + k$, while the left-hand velocity stays 1. Hence the C^1 requirement forces $\boxed{k = 0}$.

A second-order problem carries two pieces of state information. Position matching alone would admit a jump in velocity that is excluded by the problem's stated regularity; checking only the differential equation away from the switch would miss it.