

Question 9

Consider the nonlinear equation

$$y'' = \sqrt{|y|}, \quad y(0) = 0, \quad y'(0) = 0.$$

For a real parameter $a \geq 0$, define

$$y_a(t) = \begin{cases} 0, & t \leq a, \\ (t-a)^4/144, & t > a. \end{cases}$$

A classical solution here means a real C^2 function satisfying the equation at every point of its interval.

Tasks

1. Verify the equation on each open side of $t = a$.
2. Check the value, first derivative and second derivative at the joining point. Does the equation hold there as well?
3. Verify both initial conditions for every $a \geq 0$, and show that different finite choices of a give different solutions. Identify one additional solution.
4. Sketch y_0 , y_1 and the zero solution on $[0, 3]$. Explain why continuity of the right-hand side does not justify applying the linear initial-value uniqueness theorem to this example. Does an assigned position and slope always determine a unique trajectory?

Question 9 – Solution

Strategy. Verify the join at the derivative order required by the equation, then use the family as a counterexample to an overgeneralized uniqueness claim.

Step 1: Verify away from the join. For $t < a$, both sides of the equation are zero. For $t > a$,

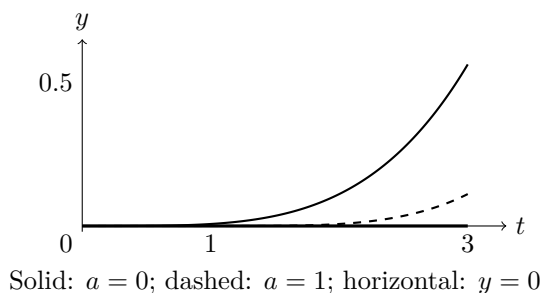
$$y'_a = \frac{(t-a)^3}{36}, \quad y''_a = \frac{(t-a)^2}{12}, \quad \sqrt{|y_a|} = \sqrt{\frac{(t-a)^4}{144}} = \frac{(t-a)^2}{12}.$$

The coefficient 144 is essential to this identity.

Step 2: Verify classical regularity at the join. The right-hand limits of y_a, y'_a, y''_a as $t \downarrow a$ are all zero, matching the values on the left. The first and second derivatives at the join are also zero, as follows directly from their difference quotients. Hence y_a is C^2 on $\mathbb{R}$.

At $t = a$, the equation reads $0 = \sqrt{|0|}$, so it holds there, not merely away from the join. There is no hidden corner in position or velocity.

Step 3: Check the common data and distinctness. Since $a \geq 0$, every family member has $y_a(0) = y'_a(0) = 0$, including $a = 0$ by the join calculation. If $a_1 < a_2$, choose $a_1 < t < a_2$. Then $y_{a_1}(t) > 0$ but $y_{a_2}(t) = 0$, proving they are different solutions. The function $\boxed{y \equiv 0}$ is another solution, different from every finite-delay member.



Step 4: State exactly what has failed. The right-hand side $\sqrt{|y|}$ is continuous but depends nonlinearly on the unknown function. This is not an equation $y'' + p(t)y' + q(t)y = g(t)$ with prescribed continuous coefficients, so that linear theorem cannot be invoked. The verified family proves that continuity alone does not ensure uniqueness for a general nonlinear second-order equation.

Two initial data determine one solution under appropriate hypotheses, not for every differential equation. This example exhibits infinitely many solutions; it does not claim to classify all solutions of the nonlinear equation.