

Question 8

Suppose both $y_1(t) = 1$ and $y_2(t) = t^2$ solve an unknown normalized homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on $(0, \infty)$.

Tasks

1. Determine p and q directly from the two supplied solutions. Check your result by substitution.
2. Can any normalized equation of this form, with continuous p, q on an interval containing zero, have both supplied functions as solutions there? Prove your answer without a determinant formula.
3. Multiply the equation from Task 1 by t . Show that the undivided equation admits $y = C_1 + C_2t^2$ on all of $\mathbb{R}$. Determine all members of this family satisfying $y(0) = a$, $y'(0) = b$.
4. Sketch $y = t^2$ and $y = 2t^2$ near zero. Explain how distinct solutions can share the same tangent there without contradicting the regular linear initial-value theorem.

Question 8 – Solution

Strategy. Insert simple known solutions to recover coefficients, then inspect what changes when the leading coefficient vanishes.

Step 1: Recover the equation on its stated interval. Inserting $y_1 = 1$ gives $q(t) = 0$. Inserting $y_2 = t^2$ then gives $2 + 2tp(t) = 0$, so

$$\boxed{p(t) = -1/t, \quad q(t) = 0}, \quad \boxed{y'' - \frac{1}{t}y' = 0 \quad (t > 0)}.$$

The constant solution has both derivatives zero, and $2 - (1/t)(2t) = 0$ verifies the quadratic solution.

Step 2: Rule out a regular equation through zero. At $t = 0$, the function t^2 has value 0, first derivative 0 and second derivative 2. In any normalized equation with finite coefficients at zero it would require

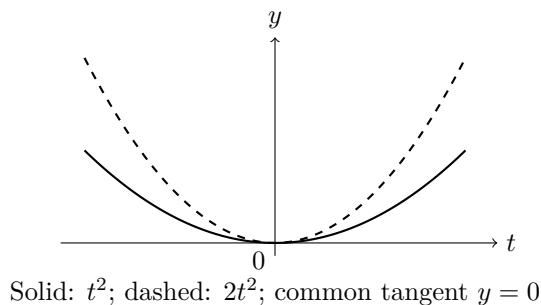
$$2 + p(0) \cdot 0 + q(0) \cdot 0 = 0,$$

which is impossible. Thus no continuous-coefficient normalized equation on an interval containing zero can have both supplied functions as solutions. The obstruction is already visible from the quadratic alone.

Step 3: Retain the degenerate leading coefficient. The undivided equation is $ty'' - y' = 0$. For $y = C_1 + C_2t^2$, its left-hand side is $t(2C_2) - 2C_2t = 0$, including at zero. The family has $y(0) = C_1$, $y'(0) = 0$, so

$$\boxed{b = 0 : y = a + C_2t^2 \ (C_2 \in \mathbb{R})}, \quad \boxed{b \neq 0 : \text{no member}}.$$

Indeed, the undivided equation itself at zero forces $y'(0) = 0$, so nonzero b permits no C^2 solution at all.



Step 4: Explain the shared tangent. The two plotted functions share value 0 and slope 0 at zero but have different curvatures. The coefficient of y'' vanishes there, so the equation does not determine the second derivative from those data. Its normalized coefficient $-1/t$ is not continuous at the initial point. The regular linear theorem therefore does not apply, and the multiple zero-data solutions do not contradict it.