

Question 7

Consider $y'' + 4y = 0$ with $y(0) = 1$, $y'(0) = 2$. Define

$$E(t) = (y'(t))^2 + 4(y(t))^2.$$

For direct verification you may use the candidate $y(t) = \cos(2t) + \sin(2t)$.

Tasks

1. Prove that E is constant for every solution and calculate its value for the stated data.
2. Deduce a bound on $|y(t)|$ without first substituting a solution formula. Verify the candidate and determine a time when the bound is attained.
3. In coordinates (y, v) with $v = y'$, sketch the curve forced by the energy identity, mark the initial state and indicate the direction of motion there.
4. Prove uniqueness directly: if two solutions share both initial data, apply the same identity to their difference. Explain why knowing the energy alone does not specify a unique solution.

Question 7 – Solution

Strategy. Differentiate a sum of nonnegative terms, then use its conservation both to bound a solution and to eliminate a zero-data difference.

Step 1: Derive the conserved quantity. For any solution,

$$E' = 2y'y'' + 8yy' = 2y'(y'' + 4y) = 0.$$

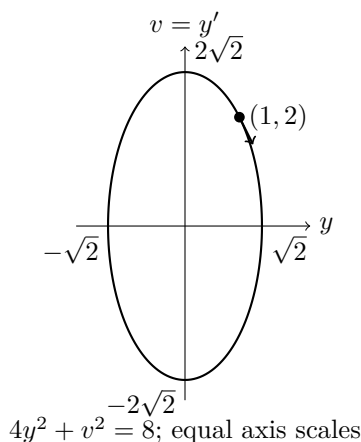
Thus $E(t) = E(0)$ throughout its interval. The prescribed data give

$$E = 2^2 + 4(1)^2 = 8.$$

Step 2: Obtain and attain the displacement bound. Since $(y')^2 \geq 0$, $4y^2 \leq 8$, whence $|y| \leq \sqrt{2}$. For the supplied candidate,

$$y' = -2\sin(2t) + 2\cos(2t), \quad y'' = -4y.$$

It has the required data, and at $t = \pi/8$ it has $y = \sqrt{2}$, $y' = 0$. Therefore the bound is sharp, not just a convenient overestimate.



Step 3: Interpret the state curve. Writing $v = y'$ gives $4y^2 + v^2 = 8$, an ellipse with intercepts $y = \pm\sqrt{2}$ and $v = \pm 2\sqrt{2}$. The initial point is $(1, 2)$. Its velocity in these coordinates is

$$(y', v') = (v, -4y) = (2, -4),$$

so it moves rightward and downward there. The axes represent state variables, not time versus displacement.

Step 4: Prove uniqueness and distinguish energy from data. The difference w of two solutions with the same initial data satisfies $w'' + 4w = 0$ and $w(0) = w'(0) = 0$. Its conserved quantity is $(w')^2 + 4w^2 = 0$. Both terms are nonnegative, so $w = 0$ everywhere on their common interval.

Energy alone only fixes an ellipse. For example, $\sqrt{2}\cos(2t)$ and $\sqrt{2}\sin(2t)$ both have energy 8 but different initial states. Initial position and velocity identify a point and its motion; the scalar energy value does not replace them.