

Question 6

Let $L[y] = y'' + y$ and consider the forced equation $L[y] = t$. The functions t , $\sin t$ and $\cos t$ are supplied for direct verification.

Tasks

1. Compute $L[t]$, $L[\sin t]$ and $L[\cos t]$. Determine exactly which constants a, b, c make $at + b\sin t + c\cos t$ solve the forced equation on $\mathbb{R}$.
2. Let $u = t + \sin t$ and $v = t + \cos t$. Determine the equations satisfied by $u + v$, $u - v$ and $(u + v)/2$.
3. For arbitrary solutions u, v of $L[y] = t$, determine exactly which real weights α, β ensure $\alpha u + \beta v$ is again a solution. Explain the difference from homogeneous superposition.
4. Find a solution with $y(0) = 2$, $y'(0) = 0$. Verify it and justify uniqueness using the continuous-coefficient initial-value theorem.

Question 6 – Solution

Strategy. Apply the linear operator to the proposed combination and track the forcing as carefully as the unknown function.

Step 1: Identify the forced and zero-forcing pieces. Direct differentiation yields

$$L[t] = t, \quad L[\sin t] = 0, \quad L[\cos t] = 0.$$

Linearity then gives $L[at + b \sin t + c \cos t] = at$. Equality with t for every real t requires $a = 1$; b, c are arbitrary. Hence all such candidates are

$$\boxed{y = t + b \sin t + c \cos t}.$$

The forcing fixes the coefficient of t , leaving the two homogeneous freedoms.

Step 2: Track sums, differences and averages. Because $L[u] = L[v] = t$,

$$L[u + v] = 2t, \quad L[u - v] = 0, \quad L[(u + v)/2] = t.$$

The sum is generally not a solution of the original forced equation. The difference solves the associated homogeneous equation, and the average preserves the original forcing.

Step 3: Determine the allowed weights. For any such u, v ,

$$L[\alpha u + \beta v] = (\alpha + \beta)t.$$

This equals t on $\mathbb{R}$ exactly when $\boxed{\alpha + \beta = 1}$. The weights need not be nonnegative. For homogeneous solutions, by contrast, the right-hand side remains zero for every pair of weights. Also, the zero function is not a solution of $L[y] = t$; the forced solution set does not have all the closure properties of the homogeneous one.

Step 4: Fit and verify the initial data. For the candidate family, $y(0) = c$ and $y'(0) = 1 + b$. The data give $c = 2$, $b = -1$, so

$$\boxed{y = t - \sin t + 2 \cos t}.$$

Its derivative is $1 - \cos t - 2 \sin t$, its initial value and slope are 2 and 0, and $y'' + y = t$. Since $p = 0$, $q = 1$, $g = t$ are continuous on $\mathbb{R}$, the initial-value theorem proves uniqueness there. No general method for finding particular solutions was assumed; the necessary candidate was supplied.