

Question 4

For $y'' + y = 0$, you may use the complete family

$$y(t) = A \cos t + B \sin t.$$

Consider two endpoint conditions $y(0) = 0$ and $y(L) = c$, where $L > 0$ is specified. Compare these with initial data $y(0) = 0$, $y'(0) = b$.

Tasks

1. Solve the initial-data problem for an arbitrary real b . Explain why it always determines exactly one member of the family.
2. Determine all solutions of the endpoint problem for $L = \pi/2$ and arbitrary c .
3. Determine all solutions for $L = \pi$, treating $c = 0$ and $c \neq 0$ separately. Explain why two conditions need not select one solution.
4. For $L = \pi$, $c = 0$, sketch the solutions with $B = -1, 0, 1, 2$. Explain why their common endpoint values do not violate uniqueness for initial-value problems.

Question 4 – Solution

Strategy. Translate each condition into a constraint on A, B and check whether the second constraint supplies new information.

Step 1: Use value and slope at the same point. From $y(0) = A$ and $y'(0) = B$, initial data $0, b$ give

$$y = b \sin t.$$

Both constants are fixed, for every real b . The coefficients of the normalized equation are continuous everywhere, consistent with initial-value uniqueness.

Step 2: Move the second datum to $\pi/2$. The first condition gives $A = 0$. At the other endpoint,

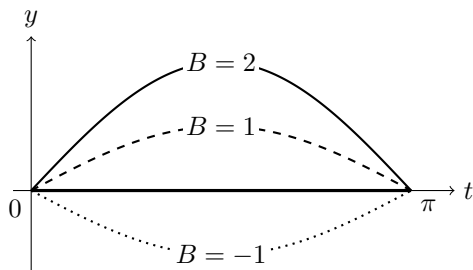
$$y(\pi/2) = B = c, \quad y = c \sin t.$$

This particular pair of endpoint conditions also determines one solution.

Step 3: Detect a redundant or incompatible condition. At $L = \pi$, every function $B \sin t$ has $y(\pi) = 0$. Consequently,

$$c = 0 : y = B \sin t \ (B \in \mathbb{R}), \quad c \neq 0 : \text{no solution}.$$

When $c = 0$, the second condition adds no restriction on B . When $c \neq 0$, it contradicts the first condition and the equation. Merely counting two stated conditions misses this distinction.



The horizontal solution is $B = 0$.

Step 4: Compare endpoint agreement with initial-data agreement. The four curves share their endpoint positions but have different initial slopes B . At $t = \pi$ their slopes are $-B$, also different. The initial-value theorem requires agreement of both value and first derivative at the same point. Agreement of values at two separate points is a different requirement, so the sketch creates no contradiction.