

Question 3

The functions e^t and e^{-t} are supplied as candidate solutions of

$$y'' - y = 0.$$

You may use this theorem: if p, q, g are continuous on an open interval I , then $y'' + py' + qy = g$ with $y(t_0) = a$, $y'(t_0) = b$, $t_0 \in I$, has exactly one solution on I .

Tasks

1. Verify that every $y = Ae^t + Be^{-t}$ solves the equation, without using a characteristic equation.
2. Determine A, B when $y(\ln 2) = 5/2$ and $y'(\ln 2) = 3/2$. Verify both data.
3. For arbitrary real data $y(\ln 2) = a$, $y'(\ln 2) = b$, solve for A, B . Show that every pair of initial data can be realized.
4. Use the theorem to prove that the displayed family contains every solution on $\mathbb{R}$. Explain why having two named constants would not be sufficient if the proposed family were $Ce^t + De^t$ instead.

Question 3 – Solution

Strategy. Check the functions, solve the two data equations, then use uniqueness to establish completeness.

Step 1: Verify the entire family. Differentiation gives

$$y' = Ae^t - Be^{-t}, \quad y'' = Ae^t + Be^{-t} = y.$$

Therefore $y'' - y = 0$ for every A, B . No root-solving method is required.

Step 2: Fit the stated data at their actual base point. Since $e^{\ln 2} = 2$ and $e^{-\ln 2} = 1/2$, the equations are

$$2A + B/2 = 5/2, \quad 2A - B/2 = 3/2.$$

Adding and subtracting yields $A = 1$, $B = 1$. Thus

$$\boxed{y = e^t + e^{-t}}.$$

Its value at $\ln 2$ is $2 + 1/2 = 5/2$, and its derivative there is $2 - 1/2 = 3/2$.

Step 3: Solve the general data problem. For arbitrary a, b the same equations give

$$4A = a + b, \quad B = a - b, \quad \boxed{A = (a + b)/4, \quad B = a - b}.$$

There is exactly one coefficient pair for each $(a, b) \in \mathbb{R}^2$. Substitution returns $y(\ln 2) = a$ and $y'(\ln 2) = b$, so the claim includes all data, not only the numerical example.

Step 4: Establish completeness, not just verification. Here $p = 0$, $q = -1$, $g = 0$ are continuous on $\mathbb{R}$. Any solution has some data a, b at $\ln 2$. Step 3 constructs a member of the displayed family with those same data; uniqueness forces agreement on $\mathbb{R}$.

By contrast, $Ce^t + De^t = (C + D)e^t$ has only one effective parameter. Every such function satisfies $y' = y$, so at the base point it can realize only data with $b = a$. Two symbols for constants do not necessarily provide two independent freedoms.