

Question 1

Consider the following differential equations for a real function $y(t)$:

$$\begin{aligned} \text{A : } (ty')' + y &= \sin t, & \text{B : } y'' + ty' + y^2 &= 0, \\ \text{C : } y'' + (y')^2 &= 0, & \text{D : } y'' + t^2y &= 0. \end{aligned}$$

An equation is linear in y if y and its derivatives occur to the first power, are not multiplied together, and have coefficients depending only on t . The terms homogeneous and nonhomogeneous here refer to linear equations with zero and nonzero forcing, respectively.

Tasks

1. Expand equation A. Classify all four equations by order and linearity; classify the linear ones by homogeneity.
2. Explain why t^2y does not destroy linearity, whereas y^2 and $(y')^2$ do. Does a zero right-hand side make B or C a homogeneous linear equation?
3. Put A in the form $y'' + p(t)y' + q(t)y = g(t)$. Find the largest open interval containing $t = 1$ on which these coefficients are continuous.
4. Suppose a C^2 solution of the undivided equation A passes through $t = 0$. Derive the necessary relation between $y(0)$ and $y'(0)$. Can $y(0) = 1$, $y'(0) = 0$ be prescribed there?

Question 1 – Solution

Strategy. Expand derivatives before classifying, and distinguish an equation from its normalized form at a zero leading coefficient.

Step 1: Expand and classify. The product rule gives $(ty')' = ty'' + y'$. Hence A is

$$ty'' + y' + y = \sin t.$$

The classification is

equation	order	type
A	2	linear, nonhomogeneous
B	2	nonlinear
C	2	nonlinear
D	2	linear, homogeneous

A is a second-order equation whose leading coefficient vanishes at one point; its order as an equation is not reclassified solely at that point.

Step 2: Identify what is being squared. In D, t^2 is a known coefficient, while y occurs to the first power. In B and C the unknown function or its derivative is squared. A zero right-hand side does not remove that nonlinearity. Under the stated convention, neither B nor C is a homogeneous linear equation.

Step 3: Normalize on a regular interval. For $t \neq 0$, division by t yields

$$y'' + \frac{1}{t}y' + \frac{1}{t}y = \frac{\sin t}{t}.$$

All three normalized coefficients are continuous on $(0, \infty)$, the largest open interval containing 1 with this property. Although $\sin t/t$ has a removable singularity at zero, $1/t$ does not.

Step 4: Test the original equation at zero. For a C^2 solution, the undivided equation at zero requires

$$0 \cdot y''(0) + y'(0) + y(0) = 0, \quad y'(0) + y(0) = 0.$$

The proposed data give $0 + 1 \neq 0$, so no such solution exists. The condition is necessary; deriving it alone is not an existence or uniqueness proof for every compatible pair. Division by the leading coefficient must not hide this compatibility requirement.