

Question 10

Use explicit Euler to integrate $y' = -2y$ from $t = 1$ backward to $t = 0$. Begin with the benchmark terminal value $y(1) = e^{-2}$ and use time increments $\Delta t = -1/4$. This is the explicit Euler formula on a decreasing time grid, not the implicit method sometimes called backward Euler.

Tasks

1. Derive the update on the decreasing grid and compute the approximation at $t = 0$.
2. Find the benchmark exact solution, calculate the reconstruction error at 0, and sketch the numerical polygon with its direction of computation.
3. Replace the supplied terminal value by $e^{-2} + \delta$. Determine how this perturbation affects the numerical reconstruction and the exact reconstruction.
4. Now suppose e^{-2} is a reported terminal measurement with unknown error at most 0.001. Find the interval of possible exact initial values. Can reducing the Euler step alone guarantee reconstruction within 0.005 of the unknown true initial value? Explain.

Question 10 – Solution

Strategy. Track the sign of the time increment and distinguish discretization error from amplification of uncertain terminal data.

Step 1: March backward explicitly. Writing $t_j = 1 - j/4$, Euler gives

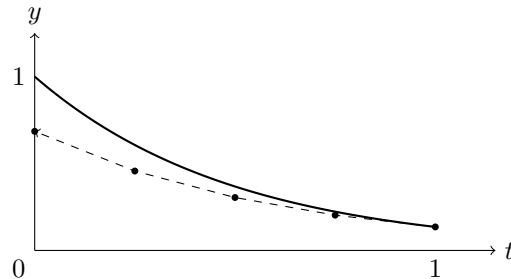
$$Y_{j+1} = Y_j - \frac{1}{4}(-2Y_j) = \frac{3}{2}Y_j.$$

After four steps,

$$Y(0) = \left(\frac{3}{2}\right)^4 e^{-2} = \frac{81}{16}e^{-2} \approx 0.685135.$$

Step 2: Compare with the exact benchmark. The exact solution is $y(t) = e^{-2t}$, giving $y(0) = 1$. Thus the numerical reconstruction error is

$$Y(0) - y(0) = \frac{81}{16}e^{-2} - 1 \approx -0.314865.$$



Solid: exact; dashed: Euler computed from $t = 1$ to $t = 0$

Each decreasing-time step amplifies the current value by 1.5. The exact amplification over a quarter-unit backward step is $e^{1/2} > 1.5$, explaining the numerical underestimate.

Step 3: Propagate a data perturbation. Linearity shows that replacing the terminal value by $e^{-2} + \delta$ changes the numerical reconstruction by

$$\left(\frac{81}{16}\right)\delta,$$

whereas exact backward evolution changes it by $e^2\delta$. With N backward steps of size magnitude $1/N$, the numerical amplification is $(1 + 2/N)^N$, which tends to e^2 , not to zero. Refinement improves the evolution calculation but does not eliminate sensitivity to data.

Step 4: Separate the uncertainty floor. The possible exact initial values form

$$[1 - 0.001e^2, 1 + 0.001e^2] \approx [0.992611, 1.007389].$$

Its half-width is about 0.007389, exceeding 0.005. No single reconstruction from the reported value can be within 0.005 of every possible true initial value in that interval. Reducing the step alone therefore cannot supply the requested guarantee; the terminal data would also need tighter uncertainty. Forward physical decay and backward data amplification describe different directions of evolution.