

Question 8

For $y' = -y$, $y(0) = 1$, let Y_h denote the Euler approximation at $t = 1$ with equal step h . Compare $h = 1/2$ and $h = 1/4$. Assume exact arithmetic. For the extrapolation argument, suppose $Y_h = y(1) + Ch + O(h^2)$ with a coefficient C independent of h as $h \rightarrow 0$.

Tasks

1. Compute $Y_{1/2}$, $Y_{1/4}$, and $R = 2Y_{1/4} - Y_{1/2}$.
2. Use the stated error expansion to explain why R cancels the leading first-order error. Relate the difference between the two Euler values to an estimate of the finer-grid error.
3. Compare the actual errors of the two Euler values and R with the exact endpoint. Is it a contradiction if R lies outside the interval between the Euler values?
4. Test whether grid agreement certifies accuracy using a different IVP, $z' = \sin^2(4\pi t)$, $z(0) = 0$. Compute both Euler endpoints on the same grids and the exact value $z(1)$, and explain the failure of the grid-difference indicator.

Question 8 – Solution

Strategy. Separate asymptotic error cancellation from a rigorous certificate: two grids can share the same sampling blind spot.

Step 1: Compute the three approximations. For decay, $Y_h = (1 - h)^{1/h}$. Hence

$$Y_{1/2} = 1/4, \quad Y_{1/4} = 81/256, \quad R = 49/128 = 0.3828125.$$

Step 2: Cancel the leading error term. The assumed expansions give

$$2Y_{h/2} - Y_h = y(1) + O(h^2), \quad Y_{h/2} - Y_h = -Ch/2 + O(h^2).$$

Thus $|Y_{h/2} - Y_h|$ estimates the leading magnitude of the finer-grid error, while the extrapolated value removes that term. This argument is asymptotic; it supplies no explicit bound for the omitted terms on these particular grids.

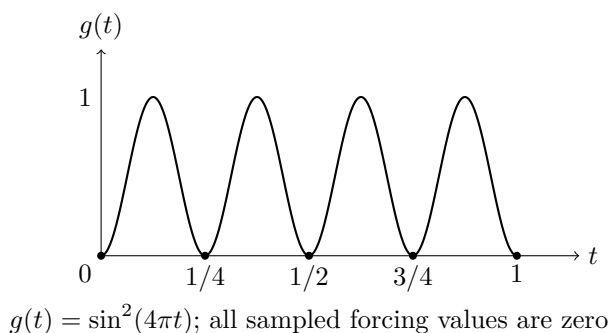
Step 3: Compare actual errors. The exact endpoint is e^{-1} . Absolute errors are

value	$Y_{1/2}$	$Y_{1/4}$	R
absolute error	0.117879	0.051473	0.014933

The finer-grid estimate is $17/256 = 0.06640625$, not its exact error. Both Euler values are below the truth, while R is above it. Extrapolation is not interpolation, so lying outside their interval is consistent with its construction and does not by itself invalidate it.

Step 4: Exhibit shared undersampling. At all left nodes of both specified grids, $\sin^2(4\pi t) = 0$. Both Euler approximations for $z(1)$ are therefore zero, and their difference is zero. But

$$z(1) = \int_0^1 \sin^2(4\pi t) dt = \frac{1}{2}.$$



The nonnegative forcing is missed between the sample nodes. Agreement of two grids does not certify accuracy; an explicit error bound or further analysis of the resolved time scales is needed.