

Question 7

For $y' = 1 - y$, $y(0) = 0$, use explicit Euler to reach $t = 1$ with N equal steps, $h = 1/N$. Assume exact arithmetic and define the signed error $E_n = Y_n - y(t_n)$. You may use $0 \leq e^{-h} - 1 + h \leq h^2/2$ for $0 < h \leq 1$.

Tasks

1. Find the exact solution and a closed formula for the Euler sequence. Determine the sign of E_n .
2. Derive an error recurrence using the one-step defect evaluated from the exact solution.
3. Prove $0 \leq E_n \leq (h/2)[1 - (1 - h)^n] \leq h/2$. Use the simpler last bound to choose an integer N guaranteeing endpoint error at most 0.005.
4. For your chosen N , calculate the actual endpoint error from the closed formulas. Explain the difference between a guaranteed bound and an observed error, and state which arithmetic assumption the proof uses.

Question 7 – Solution

Strategy. Exploit the contracting error recurrence to turn a local defect bound into a rigorous global guarantee.

Step 1: Solve both continuous and discrete problems. The exact solution is $y(t) = 1 - e^{-t}$. Euler gives $Y_{n+1} = (1 - h)Y_n + h$, so

$$\boxed{Y_n = 1 - (1 - h)^n}, \quad E_n = e^{-nh} - (1 - h)^n \geq 0.$$

For $0 < h < 1$, the sign follows from $1 - h < e^{-h}$; at $h = 1$, it follows directly for $n \geq 1$. At $n = 0$, both solutions are zero.

Step 2: Isolate the local defect. Starting Euler at the exact value at t_n , its overshoot of the exact next value is

$$d_n = y(t_n) + h[1 - y(t_n)] - y(t_n + h) = e^{-t_n}(e^{-h} - 1 + h).$$

Therefore $0 \leq d_n \leq h^2/2$, and subtracting the exact update from the numerical one gives

$$\boxed{E_{n+1} = (1 - h)E_n + d_n, \quad E_0 = 0}.$$

Step 3: Sum the propagated defects. Since $0 \leq 1 - h < 1$, iteration gives

$$0 \leq E_n \leq \frac{h^2}{2} \sum_{j=0}^{n-1} (1 - h)^j = \frac{h}{2} [1 - (1 - h)^n] \leq \frac{h}{2}.$$

Thus $1/(2N) \leq 0.005$ is sufficient. The smallest integer supplied by this simpler bound is $\boxed{N = 100}$, with $h = 0.01$. This is not a claim that no smaller N could meet the tolerance using a sharper analysis.

Step 4: Evaluate the actual error. For that grid,

$$\boxed{E_{100} = e^{-1} - (0.99)^{100} \approx 0.00184710 < 0.005}.$$

The proven bound controls the error without assuming a particular observed cancellation or leading asymptotic term. Its conservatism explains the smaller actual error. Both the recurrence and this guarantee assumed exact arithmetic; step-by-step rounding would add another error term and require a separate bound.