

**Question 6**

Consider

$$y' = y^2, \quad y(0) = 1.$$

Two algorithms use a grid with  $h = 1/4$ . Algorithm A applies explicit Euler directly to  $y$ . Algorithm B first sets  $u = 1/y$ , applies explicit Euler to the transformed equation, and then recovers  $y = 1/u$  wherever possible.

**Tasks**

1. Derive both update rules and compute their approximations at  $t = 1/4$  and  $t = 1/2$ .
2. Find the exact IVP solution and determine which algorithm is exact at these nodes. Explain why this happens here.
3. Starting from a common positive value  $y$  with  $hy < 1$ , compare the two one-step updates algebraically and find their exact difference.
4. Explain what happens to algorithm B at  $t = 1$  and why finite direct-Euler values at or beyond that time cannot establish continuation of the original IVP.

## Question 6 – Solution

**Strategy.** A nonlinear change of dependent variable preserves the exact differential equation but generally does not commute with a finite Euler step.

**Step 1: Derive and compute the algorithms.** Direct Euler gives  $Y_{n+1} = Y_n + hY_n^2$ ,  $Y_0 = 1$ . Since  $u' = -y'/y^2 = -1$ , transformed Euler gives  $U_{n+1} = U_n - h$ ,  $U_0 = 1$ . Thus

| $t$   | A: direct Euler     | B: $1/U_n$ |
|-------|---------------------|------------|
| $1/4$ | $5/4$               | $4/3$      |
| $1/2$ | $105/64 = 1.640625$ | $2$        |

**Step 2: Compare with the exact solution.** Separation gives  $y = 1/(1-t)$ , whose maximal interval containing zero is  $(-\infty, 1)$ . Algorithm B is exact at every grid node before 1, because its transformed equation has the constant slope  $-1$ : Euler integrates that affine solution exactly. Algorithm A underestimates the two displayed exact values.

**Step 3: Compare a single step.** From the same positive state, the updates are

$$A(y) = y + hy^2, \quad B(y) = \frac{1}{1/y - h} = \frac{y}{1 - hy}.$$

Their exact difference is

$$B(y) - A(y) = \frac{h^2 y^3}{1 - hy} > 0 \quad (hy < 1).$$

For fixed  $y$ , the discrepancy is of order  $h^2$  as  $h \rightarrow 0$ , although both are Euler approximations in their chosen coordinates. The nonlinear reciprocal map changes the finite-step rule.

**Step 4: Respect the original blow-up.** At  $t = 1$ , algorithm B has  $U_4 = 0$ , so its inverse is undefined, matching the exact pole. The direct recurrence uses only addition and multiplication and gives a finite real value after any fixed finite number of exact-arithmetic steps. Those values are numerical iterates, not proof of a classical solution through the pole.

The original solution tends to  $+\infty$  as  $t \uparrow 1$ , so it cannot extend continuously across that time. Beyond 1, an inverse value from negative  $U_n$  would likewise belong to a disconnected formula branch, not a continuation of this IVP.