

Question 3

Consider a decaying quantity satisfying

$$y' = -4y, \quad y(0) = 1.$$

Apply explicit Euler with a constant step $h > 0$ for arbitrarily many steps.

Tasks

1. Derive the numerical solution Y_n and determine exactly which steps give $Y_n \rightarrow 0$ as $n \rightarrow \infty$.
2. Determine the stricter step condition that keeps every numerical value nonnegative and the sequence nonincreasing. Discuss $h = 1/4$ and $h = 1/2$ separately.
3. Compute enough values for $h = 0.4$ and $h = 0.6$ to show their contrasting behavior, and compare them graphically with the exact decay.
4. Explain why decreasing the physical solution's magnitude does not by itself ensure decay of an Euler approximation, and distinguish stability from faithful positivity.

Question 3 – Solution

Strategy. Analyze the discrete amplification factor rather than transferring the differential equation's stability directly to its numerical update.

Step 1: Solve the recurrence. The update is $Y_{n+1} = (1 - 4h)Y_n$, so $Y_n = (1 - 4h)^n$. It tends to zero exactly when

$$|1 - 4h| < 1, \quad 0 < h < 1/2.$$

Step 2: Impose positivity and monotonicity. All values are nonnegative and nonincreasing exactly when $0 \leq 1 - 4h < 1$, or

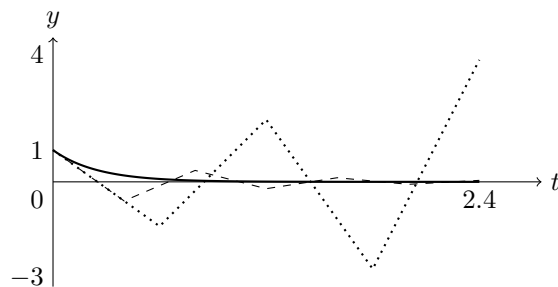
$$0 < h \leq 1/4.$$

At $h = 1/4$, the approximation becomes zero after one step, although e^{-4t} is strictly positive at every finite time. At $h = 1/2$, $Y_n = (-1)^n$ oscillates without decay. For $h > 1/2$, its magnitude grows without bound.

Step 3: Compare two oscillating sequences.

n	0	1	2	3	4
$h = 0.4$	1	-0.6	0.36	-0.216	0.1296
$h = 0.6$	1	-1.4	1.96	-2.744	3.8416

Each row is plotted at its own times $t_n = nh$, not at common node indices interpreted as time.



Solid: exact; dashed: $h = 0.4$; dotted: $h = 0.6$

Step 4: Interpret numerical stability. The exact solution $y = e^{-4t}$ decays smoothly. Euler replaces its per-step factor e^{-4h} by $1 - 4h$, which can be negative or have magnitude exceeding one. At $h = 0.4$ the method is asymptotically decaying but produces unphysical sign changes; at $h = 0.6$ it also loses decay. Stability controls long-run amplification, while positivity is an additional property with a stricter step requirement.