

Question 2

For $y' = y$, $y(0) = 1$, apply explicit Euler on $[0, 1]$ with N equal steps and $h = 1/N$. Let Y_N denote its endpoint value. You may use $\ln(1 + h) = h - h^2/2 + O(h^3)$ and $e^z = 1 + z + O(z^2)$ as the arguments tend to zero.

Tasks

1. Derive a closed formula for Y_N and prove that it underestimates the exact endpoint for every positive integer N .
2. Compute the one-step defect when Euler starts from an exact value $y(t_n)$. Show that this defect is of order h^2 .
3. Derive the leading endpoint error as $h \rightarrow 0$ and explain why its order differs from that of the one-step defect.
4. Calculate endpoint values and absolute errors for $N = 2, 4, 8$. Explain what error reduction is expected when N doubles and what the computation cost does.

Question 2 – Solution

Strategy. Solve the discrete recurrence exactly, then compare its logarithm with the continuous exponential growth.

Step 1: Find and bound the endpoint. Euler gives $Y_{n+1} = (1 + h)Y_n$, so

$$Y_N = (1 + 1/N)^N.$$

The exact solution is e^t . Since $\ln(1 + h) < h$ for $h > 0$, $N \ln(1 + 1/N) < 1$ and hence $Y_N < e$. This is an underestimate for every grid, not merely for the three grids below.

Step 2: Define the one-step defect. Starting with the exact value e^{t_n} , the difference between the exact next value and one Euler step is

$$d_n = e^{t_n+h} - e^{t_n}(1 + h) = e^{t_n}(e^h - 1 - h) = \frac{1}{2}e^{t_n}h^2 + O(h^3).$$

On $0 \leq t_n \leq 1$, this is uniformly $O(h^2)$. Here defect means an error in solution value; dividing it by h would use a different convention.

Step 3: Derive the accumulated error. Because $Nh = 1$,

$$\ln Y_N = \frac{\ln(1 + h)}{h} = 1 - \frac{h}{2} + O(h^2).$$

Exponentiating gives $Y_N = e(1 - h/2 + O(h^2))$, so

$$e - Y_N = \frac{e}{2}h + O(h^2).$$

There are $N = 1/h$ steps over the fixed interval, and their defects propagate through later updates. An $O(h^2)$ one-step defect therefore need not yield an $O(h^2)$ endpoint error; here the accumulated error is first order.

Step 4: Compare the grids.

N	Y_N	$e - Y_N$
2	2.250000	0.468282
4	2.441406	0.276876
8	2.565785	0.152497

The errors approach a factor-of-two reduction under each doubling of N , because their leading term is proportional to h . Exact halving is not promised on coarse grids. The number of slope evaluations doubles, exposing the first-order accuracy versus work tradeoff.