

**Question 9**

Let

$$V(y) = \frac{1}{4}(y^2 - 1)^2, \quad y' = -V'(y).$$

Think of  $V$  as an energy function; no physical units are assumed.

**Tasks**

1. Derive the autonomous equation explicitly, find its equilibria, and classify them using the signs of the right-hand side.
2. Compute  $dV(y(t))/dt$  along a solution. Identify exactly when this derivative vanishes.
3. Sketch  $V$  and relate its local minima and maximum to equilibrium stability. Explain why nonincreasing energy alone does not imply convergence to a global minimum for every initial value.
4. Determine the forward limit for every real initial value and justify global forward existence. State the basins of the attracting equilibria.

## Question 9 – Solution

**Strategy.** Combine the energy identity with the phase line; energy decrease is informative but does not by itself choose a limiting equilibrium.

**Step 1: Derive the flow and classify its zeros.** Since  $V'(y) = y(y^2 - 1)$ ,

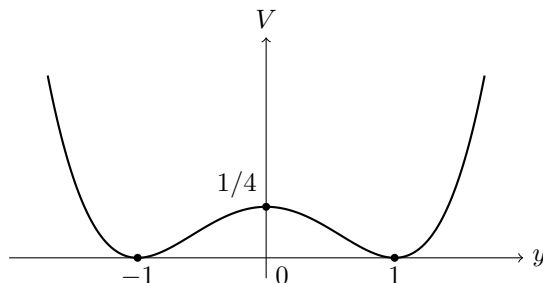
$$y' = y - y^3 = y(1 - y^2).$$

Its zeros are  $-1, 0, 1$ . Its signs on the successive intervals are  $+, -, +, -$ , so  $-1$  and  $1$  are asymptotically stable and  $0$  is unstable.

**Step 2: Differentiate the energy.** The chain rule gives

$$\frac{dV}{dt} = V'(y)y' = -[V'(y)]^2 = -(y - y^3)^2 \leq 0.$$

The derivative vanishes exactly at  $y = -1, 0, 1$ . On any nonconstant trajectory it is strictly negative, since uniqueness prevents meeting an equilibrium at a finite time.



Energy minima:  $\pm 1$ ; local maximum:  $0$

**Step 3: Interpret the energy landscape.** The points  $\pm 1$  are minima with  $V = 0$ ;  $0$  is a local maximum with  $V = 1/4$ . Since the flow is down the energy gradient, the minima attract and the maximum repels. Nevertheless, the initial value  $y(0) = 0$  stays at that maximum forever, with constant energy. Nonincreasing energy does not force every solution to reach a global minimum, nor distinguish which minimum will be selected.

**Step 4: Determine the basins and future limits.** The decreasing-energy bound confines a trajectory to  $\{y : V(y) \leq V(y(0))\}$ , a bounded set because  $V(y) \rightarrow \infty$  as  $|y| \rightarrow \infty$ . Smoothness then ensures global forward continuation. Phase-line trapping gives bounded monotonic motion for every nonconstant solution. Its limit must be a zero of  $y - y^3$ , and it cannot cross zero. Therefore

$$\lim_{t \rightarrow \infty} y(t) = \begin{cases} -1, & y(0) < 0, \\ 0, & y(0) = 0, \\ 1, & y(0) > 0. \end{cases}$$

The basins are  $(-\infty, 0)$  for  $-1$  and  $(0, \infty)$  for  $1$ .