

Question 8

Consider the autonomous equation with its stated domain:

$$y' = \frac{y-1}{y+2}, \quad y \neq -2.$$

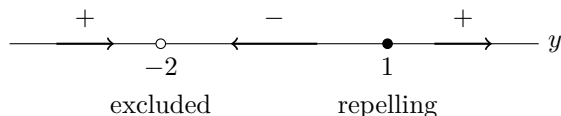
Tasks

1. Identify all equilibria and construct a phase line that distinguishes an excluded level from a zero of the right-hand side.
2. Classify each equilibrium and determine the direction of motion in every component cut by the equilibrium and the excluded level.
3. For $y(0) = 0$, derive an implicit solution relation and find the exact maximal forward time range.
4. Determine the limiting value and slope at its finite endpoint. Explain why approaching a finite level does not make that level an attracting equilibrium and why multiplying by $y + 2$ does not remove the obstruction for this original equation.

Question 8 – Solution

Strategy. A denominator zero splits the domain but is not an equilibrium. Test whether the selected trajectory reaches that boundary in finite time.

Step 1: Separate zeros from exclusions. The only equilibrium is $y = 1$. The value $y = -2$ is outside the equation's domain. The signs of f are positive on $(-\infty, -2)$, negative on $(-2, 1)$, and positive on $(1, \infty)$.



Thus 1 is unstable: arrows point away on both sides. Also $f'(y) = 3/(y+2)^2$, so $f'(1) = 1/3 > 0$ confirms the classification.

Step 2: Integrate the selected branch. Starting at 0, the solution decreases within $(-2, 1)$. Separation gives

$$dt = \frac{y+2}{y-1} dy, \quad t = y + 3 \ln(1-y).$$

The constant is zero from $y(0) = 0$. On $-2 < y \leq 0$, the derivative of the right side with respect to y is $(y+2)/(y-1) < 0$. It is therefore invertible along this branch, and implicit differentiation verifies the original equation.

Step 3: Determine the endpoint. As $y \downarrow -2$, the time tends to

$$T = -2 + 3 \ln 3 \approx 1.296, \quad 0 \leq t < T.$$

This is the maximal forward range: the inverse exists until T , but at T its limiting value is excluded. Within the branch,

$$y(t) \rightarrow -2^+, \quad y'(t) = \frac{y-1}{y+2} \rightarrow -\infty.$$

Step 4: Interpret the domain boundary. An attracting equilibrium must itself define a constant solution of the given equation. Here -2 does not, and the selected solution reaches the boundary in finite time with unbounded slope rather than converging to a valid equilibrium as $t \rightarrow \infty$.

Multiplication gives $(y+2)y' = y-1$, equivalent to the original only when $y \neq -2$. At $y = -2$ it would demand $0 = -3$ for any finite derivative. Thus even the multiplied relation supplies no differentiable continuation through this boundary.