

Question 5

Consider the one-parameter family

$$y' = \mu - y^2, \quad \mu \in \mathbb{R}.$$

Tasks

1. Find and classify all equilibria for $\mu < 0$, $\mu = 0$, and $\mu > 0$. State the one-sided behavior at any semistable case.
2. Draw the equilibrium branches in the (μ, y) -plane, distinguishing attracting and repelling branches and marking the exceptional parameter value.
3. Solve the IVP $y(0) = 0$ for each parameter regime and determine its forward behavior and maximal forward time range.
4. Explain how the number and stability of equilibria change at $\mu = 0$, and why the derivative test alone does not settle that parameter value.

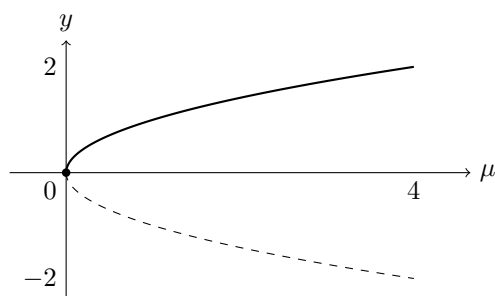
Question 5 – Solution

Strategy. Compare a horizontal parameter with y^2 , and treat the double zero separately from the two simple roots.

Step 1: Classify the parameter regimes. If $\mu < 0$, there are no equilibria and $y' < 0$ everywhere. If $\mu = 0$, the only equilibrium is 0: $-y^2 < 0$ on both sides, so it attracts from above and repels from below. If $\mu > 0$, the equilibria are

$$y = -\sqrt{\mu} \text{ (unstable),} \quad y = \sqrt{\mu} \text{ (asymptotically stable)}.$$

Indeed $\mu - y^2$ is positive between the roots and negative outside them. The derivative $f_y = -2y$ confirms the classifications at the simple roots.



Solid: attracting; dashed: repelling; origin: semistable

Step 2: Solve the selected IVP. Separation and $y(0) = 0$ give the following cases, with $a > 0$ as indicated:

parameter	$y(t)$	forward time range
$\mu > 0, a = \sqrt{\mu}$	$a(1 - e^{-2at})/(1 + e^{-2at})$	$0 \leq t < \infty$
$\mu = 0$	0	$0 \leq t < \infty$
$\mu < 0, a = \sqrt{-\mu}$	$-a \tan(at)$	$0 \leq t < \pi/(2a)$

For $\mu > 0$, integration on $-a < y < a$ gives $\ln((a + y)/(a - y)) = 2at$. For the last case, integration gives $\arctan(y/a) = -at$ on the initial branch. Differentiating the displayed expressions verifies $y' = \mu - y^2$ and their zero initial values.

Step 3: Describe the future. For $\mu > 0$, the selected solution increases to $\sqrt{\mu}$; at $\mu = 0$ it stays at zero; for $\mu < 0$ it decreases to $-\infty$ as $t \uparrow \pi/(2a)$, so no finite continuous extension exists.

Step 4: Interpret the branch merger. As μ decreases to zero, the attracting and repelling equilibria merge into a semistable double zero; for negative μ neither remains. At the merger $f_y(0) = 0$, so the derivative test is inconclusive. The sign of $-y^2$ supplies the one-sided classification.