

Question 3

Two proposed relaxation laws are

$$\text{A: } y' = -y^3, \quad \text{B: } y' = y^3.$$

Both have an equilibrium at 0 and derivative of the right-hand side equal to zero there.

Tasks

1. Classify 0 for each law using the sign of the right-hand side. Explain why their identical derivative-test data do not imply identical stability.
2. Solve both IVPs with $y(0) = a$, including $a = 0$, and specify their maximal forward time ranges.
3. For law A, prove stability directly from the solution and determine its large-time decay for $a \neq 0$.
4. For law B, use the solution to prove instability and find the blow-up time for $a \neq 0$. Distinguish leaving a small neighborhood from blowing up.

Question 3 – Solution

Strategy. When the linear term vanishes, retain the nonlinear term. Separation gives both a stability proof and the actual time scales.

Step 1: Classify from signs. For A, $-y^3$ is positive below 0 and negative above, so trajectories move toward 0. For B the signs reverse, so trajectories move away. Thus 0 is asymptotically stable for A and unstable for B. The shared value $f'(0) = 0$ is an inconclusive test, not evidence of neutral stability.

Step 2: Solve with the initial sign preserved. For $a \neq 0$, differentiating y^{-2} gives 2 for A and -2 for B. Applying the initial value yields

$$y_A(t) = \frac{a}{\sqrt{1+2a^2t}}, \quad t \geq 0,$$

$$y_B(t) = \frac{a}{\sqrt{1-2a^2t}}, \quad 0 \leq t < \frac{1}{2a^2}.$$

The full maximal intervals containing 0 are $(-1/(2a^2), \infty)$ and $(-\infty, 1/(2a^2))$, respectively. Direct differentiation verifies the equations. If $a = 0$, the unique solution of either smooth equation is zero for all real time.

Step 3: Prove stability and attraction for A. For $t \geq 0$, $|y_A(t)| \leq |a|$. Given $\varepsilon > 0$, choosing $\delta = \varepsilon$ ensures $|a| < \delta$ implies $|y_A(t)| < \varepsilon$ for all future time. Also $y_A(t) \rightarrow 0$ for every a , so the attraction is global. More precisely,

$$\sqrt{2t}|y_A(t)| \rightarrow 1 \quad (a \neq 0).$$

The decay is algebraic rather than exponential.

Step 4: Prove instability for B. Choose any fixed $\varepsilon > 0$ and any nonzero $|a| < \varepsilon$. The magnitude first reaches ε at

$$t_\varepsilon = \frac{1}{2a^2} - \frac{1}{2\varepsilon^2} > 0,$$

then exceeds it. Arbitrarily small nonzero perturbations therefore leave that neighborhood. Blow-up occurs later, at $1/(2a^2)$, with the sign of a . Instability only requires departure from a prescribed neighborhood; it does not require an infinite solution value.