

Question 6

An inverted conical tank has height 4 m and top radius 2 m. It initially contains water to the brim. Let $h(t)$ be water depth above the outlet, with t in seconds. Assume horizontal water surfaces and similar conical cross-sections. The outflow rate is prescribed by

$$Q(h) = \kappa\sqrt{h}, \quad \kappa = 0.05 \text{ m}^{5/2}/\text{s},$$

while $h > 0$. There is no inflow. Ignore changes in the discharge coefficient and stop outflow once the tank is empty.

Tasks

1. Derive the volume as a function of depth and use volume conservation to form an IVP for h .
2. Solve for the depth until emptying and find the emptying time.
3. Find the fractions of the emptying time needed to reach half the original depth and half the original volume. Explain why these are different.
4. Sketch the physical depth, including the empty state. Decide whether depth and volume have continuously differentiable extensions through emptying, and explain the scope of the divided depth equation.

Question 6 – Solution

Strategy. The water surface area changes with depth. Convert a volume-loss law into a depth equation before separating variables.

Step 1: Use the conical geometry. Similarity gives $r = h/2$, so $V = \pi h^3/12$ and

$$\frac{\pi h^2}{4} h' = -0.05\sqrt{h}, \quad h' = -\frac{0.20}{\pi} h^{-3/2}, \quad h(0) = 4.$$

The first equality has units m^3/s . Division by h^2 is valid only while water remains.

Step 2: Integrate to emptying. Integrating $h^{3/2} dh = -(0.20/\pi) dt$ gives $h^{5/2} = 32 - t/(2\pi)$, hence

$$h(t) = (32 - t/(2\pi))^{2/5}, \quad 0 \leq t < t_e, \quad t_e = 64\pi \approx 201.06 \text{ s}.$$

Equivalently, $h = 4(1 - t/t_e)^{2/5}$. Differentiation verifies the depth equation; the physical depth is set to zero for $t \geq t_e$.

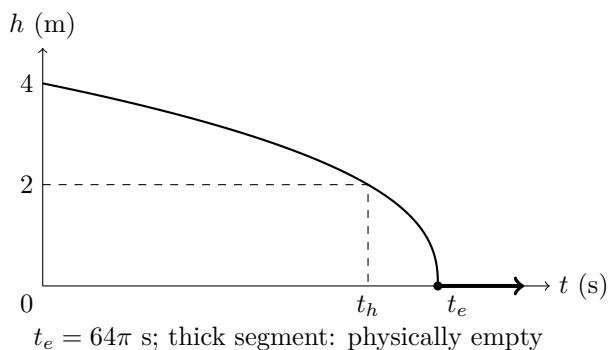
Step 3: Compare depth and volume targets. At $h = 2$ m, $h/4 = 1/2$, giving

$$t_h/t_e = 1 - 2^{-5/2} \approx 0.8232.$$

At half volume, $(h/4)^3 = 1/2$, so

$$t_V/t_e = 1 - 2^{-5/6} \approx 0.4388.$$

Half depth leaves only one eighth of the original volume, so it takes substantially longer than losing half the volume.



Step 4: Interpret the endpoint. As $t \uparrow t_e$, $h' \rightarrow -\infty$, so the zero continuation of depth is continuous but not differentiable at emptying. However, $V' = -0.05\sqrt{h} \rightarrow 0$, agreeing with $V' = 0$ afterward; volume has a continuously differentiable zero continuation. The divided depth equation is a wet-tank model and is undefined at $h = 0$. It must not be continued algebraically into a fictitious refilling branch.