

Question 3

A specimen contains a stable material of unknown mass B and an active material of mass $R(t)$ that disappears from the measured specimen at a rate proportional to R . Assume the stable material is unaffected and all disappearance products leave the measured mass. Thus the measured mass is $M(t) = B + R(t)$. Measurements, treated as exact, are

$$M(0) = 100 \text{ g}, \quad M(1) = 70 \text{ g}, \quad M(2) = 55 \text{ g},$$

where t is in days. Assume $B \geq 0$, $R(0) > 0$, and a positive constant disappearance coefficient.

Tasks

1. Formulate the active-mass IVP and the resulting measurement model.
2. Determine uniquely the stable mass, initial active mass, and disappearance coefficient from the data.
3. Find the active material's half-life and the time at which only 5 g of active material remain.
4. A technician fits exponential decay directly to the total mass using the first two measurements. Test that model against the third measurement and explain the modeling error.

Question 3 – Solution

Strategy. Separate the unchanging baseline from the decaying component. Differences of equally spaced measurements remove the baseline.

Step 1: Write the component and observation models. The active mass satisfies $R' = -kR$, where $k > 0$ has units day^{-1} . If $R(0) = R_0$, then

$$R = R_0 e^{-kt}, \quad M = B + R_0 e^{-kt}.$$

The stable material contributes to the measurement but not to the loss rate.

Step 2: Identify the parameters. Let $r = e^{-k}$, where one day is the time step, so $0 < r < 1$. The measured successive differences satisfy

$$-30 = R_0(r - 1), \quad -15 = R_0 r(r - 1).$$

Their ratio gives $r = 1/2$. Therefore

$$\boxed{R_0 = 60 \text{ g}, \quad B = 40 \text{ g}, \quad k = \ln 2 \text{ day}^{-1}}, \quad \boxed{M(t) = 40 + 60 2^{-t}}.$$

These values reproduce 100, 70, 55 g. They are unique under the assumptions because the nonzero first difference fixes r , then R_0 , then B . Also $R' = -kR$ verifies the rate law.

Step 3: Compute active-mass times. The active half-life is $\ln 2/k = \boxed{1 \text{ day}}$. The condition $R = 5$ gives $60 2^{-t} = 5$, hence

$$\boxed{t = \frac{\ln 12}{\ln 2} \approx 3.585 \text{ days}}.$$

The active mass is below 5 g after that time. The total mass then is 45 g, not 5 g.

Step 4: Test the incorrect observation model. A single exponential through the first two total-mass measurements is $\widetilde{M}(t) = 100(0.7)^t$. It predicts $\widetilde{M}(2) = 49$ g, contradicting the measured 55 g. It wrongly assigns the stable component the same disappearance law as the active material and predicts a zero limiting total mass instead of 40 g.

The model is intended for $t \geq 0$. Its inference depends on exact data, constant k , and a truly constant baseline; noisy data would require an estimation method rather than exact identification from three values.