

Question 9

Define a real function on the entire real line by

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Consider $y' = f(x)$ with $y(-1) = 0$. Require a classical solution: y is continuously differentiable on an open interval and the equation holds at every point.

Tasks

1. Find the IVP solution on $x < 0$ and its maximal interval containing -1 .
2. Compute the finite left-hand limits of y and y' at 0.
3. Find the continuous function on $\mathbb{R}$ that agrees with this solution on $x < 0$ and solves the differential equation on $x > 0$. Decide whether it is a classical extension through 0.
4. Explain why this example does not contradict a continuation theorem requiring a continuous right-hand side. Would changing only $f(0)$ repair the obstruction?

Question 9 – Solution

Strategy. A formula obtained by integrating on either side of a jump need not be differentiable at the joining point.

Step 1: Solve on the initial component. For $x < 0$, $y' = -1$, so the initial condition gives

$$\boxed{y = -x - 1}.$$

It solves the equation at every negative x . Its maximal classical interval containing -1 is $\boxed{(-\infty, 0)}$, as the join test below shows.

Step 2: Compute the finite endpoint limits. As $x \uparrow 0$, $y(x) \rightarrow -1$ and $y'(x) \rightarrow -1$. Neither the solution nor its derivative blows up. The obstruction will be the change of slope imposed immediately to the right.

Step 3: Test the unique continuous gluing. On $x > 0$, any solution is $x + C$. Continuity with the left branch forces $C = -1$ and the value $y(0) = -1$. The only continuous gluing is therefore

$$Y(x) = |x| - 1.$$

Its left difference quotient at 0 is -1 and its right difference quotient is 1 . Consequently $Y'(0)$ does not exist. It is not a classical solution of the equation on any interval containing 0.

Any larger interval extending the negative-half-line solution would include points on both sides of 0 and would require this same gluing. Thus no classical extension is possible, proving the claimed maximality despite the finite limits.

Step 4: Identify the missing hypothesis. Here f is defined everywhere but is discontinuous at 0. A continuation result that requires continuity of the right-hand side near the endpoint cannot be applied. Domain membership and bounded values alone are insufficient when that hypothesis fails.

Changing $f(0)$ to any single real value cannot reconcile the unequal one-sided derivatives forced by $f = -1$ to the left and $f = 1$ to the right. The same corner remains. The function Y solves the equation away from the join, but the question requires the equation at every point of a classical interval.