

Question 8

Consider the equation in its original, undivided form:

$$xy' = 2y, \quad y(1) = 1.$$

It is defined for all real x , including $x = 0$. A classical solution must be continuously differentiable and satisfy this equation at every point of its open interval.

Tasks

1. Solve on $x > 0$ and explain why dividing by x cannot by itself settle continuation through 0 for the original equation.
2. Find every global classical solution of the original IVP.
3. Check both the differential equation and continuous differentiability at 0 for your full family.
4. Determine the maximal interval of each such solution. Explain how uniqueness on $x > 0$ can coexist with nonunique global continuation, and why this differs from an equation explicitly defined only for $x \neq 0$.

Question 8 – Solution

Strategy. The original equation remains meaningful at the vanishing leading coefficient. Solve on the two half-lines and test the join in that original equation.

Step 1: Solve where division is allowed. For $x \neq 0$, the equation gives $(y/x^2)' = 0$. The initial condition therefore forces $y = x^2$ throughout $x > 0$. On $x < 0$, the general solution is $y = Cx^2$, with an independent real constant C .

The divided equation $y' = 2y/x$ has no meaning at 0, but the original equation does: it requires $y(0) = 0$. A singularity introduced by division is not automatically an obstruction for the original problem.

Step 2: List all global solutions. Every global candidate must have the form

$$y_C(x) = \begin{cases} Cx^2, & x < 0, \\ 0, & x = 0, \\ x^2, & x > 0, \end{cases} \quad C \in \mathbb{R}.$$

The half-line solutions and the equation at 0 leave no other possible global classical solutions.

Step 3: Verify the join. For every finite C , y_C is continuous at 0. The difference quotient there is Ch for $h < 0$ and h for $h > 0$, so $y'_C(0) = 0$. Moreover,

$$y'_C(x) = 2Cx \quad (x < 0), \quad y'_C(x) = 2x \quad (x > 0)$$

tends to 0 from both sides. Thus y_C is continuously differentiable. The identity $xy'_C = 2y_C$ holds on each half-line and reduces to $0 = 0$ at 0. Also $y_C(1) = 1$.

Step 4: Resolve maximality and uniqueness. Each y_C has maximal interval $\boxed{\mathbb{R}}$. The unique positive-half-line solution extends through 0 in infinitely many ways because the original equation loses its derivative term there. The usual normal-form uniqueness theorem cannot cross that degeneracy.

If the problem had instead specified $y' = 2y/x$ only for $x \neq 0$, its IVP interval containing 1 would be $(0, \infty)$, even for the polynomial formula. Which equation and domain were actually given matters.