

**Question 7**

Let  $p$  be any real number. Consider the family of IVPs

$$y' = y(1 - y), \quad y(0) = p.$$

Focus on intervals of validity; a full equilibrium or phase-line analysis is not required.

**Tasks**

1. Derive a solution formula in terms of  $p$ , preserving the cases lost by division in separation.
2. Classify the maximal interval containing 0 for every real  $p$ . Give any finite endpoint explicitly.
3. Determine the direction and sign of blow-up whenever such an endpoint exists.
4. Verify the formula and explain why the interval depends on  $p$  even though the differential equation is smooth on the whole  $(x, y)$ -plane.

## Question 7 – Solution

**Strategy.** A denominator zero is relevant only in the connected component containing the initial point. Preserve constant solutions before separating.

**Step 1: Obtain a parameter formula.** The constant solutions are  $y = 0, 1$ . Otherwise separation gives  $\ln |y/(1-y)| = x + C$ . Applying the initial value yields a formula that also includes both constants:

$$y(x) = \frac{pe^x}{1 - p + pe^x}.$$

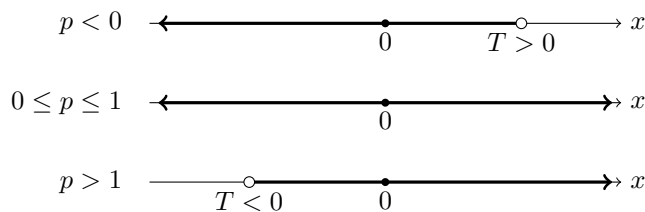
Its denominator  $D(x)$  satisfies  $D(0) = 1$ . A zero requires  $e^x = (p-1)/p > 0$ , which occurs exactly when  $p < 0$  or  $p > 1$ .

**Step 2: Select the component containing zero.** When it exists, set  $T = \ln((p-1)/p)$ . The complete classification is

| initial value     | endpoint location  | maximal interval |
|-------------------|--------------------|------------------|
| $p < 0$           | $T > 0$            | $(-\infty, T)$   |
| $0 \leq p \leq 1$ | no finite endpoint | $\mathbb{R}$     |
| $p > 1$           | $T < 0$            | $(T, \infty)$    |

For  $0 < p < 1$ , both terms of  $D = 1 - p + pe^x$  are positive. The cases  $p = 0, 1$  are global constants.

**Step 3: Determine the endpoint behavior.** If  $p < 0$ , then  $D \downarrow 0^+$  as  $x \uparrow T$ , while  $pe^x \rightarrow p - 1 < 0$ , so  $y \rightarrow -\infty$ . If  $p > 1$ , then  $D \downarrow 0^+$  as  $x \downarrow T$ , while  $pe^x \rightarrow p - 1 > 0$ , so  $y \rightarrow +\infty$ . Neither pole is removable.



Interval diagrams; endpoint positions schematic

**Step 4: Verify and interpret.** With  $N = pe^x$ ,  $N' = D' = N$ , and

$$y' = \frac{N(D - N)}{D^2} = y(1 - y), \quad y(0) = p.$$

Smoothness gives local existence and uniqueness. It does not prevent the finite-time blow-up found for  $p \notin [0, 1]$ ; the initial value determines whether the selected solution encounters a pole.