

Question 6

Consider the IVP

$$y' = 3|y|^{2/3}, \quad y(0) = 0.$$

For finite real numbers $a \leq 0 \leq b$, define

$$Y_{a,b}(x) = \begin{cases} (x-a)^3, & x < a, \\ 0, & a \leq x \leq b, \\ (x-b)^3, & x > b. \end{cases}$$

Tasks

1. Verify the equation and continuous differentiability of $Y_{a,b}$, including both joining points.
2. Give two distinct global solutions satisfying the initial condition and determine their maximal intervals.
3. Explain why a restriction of $Y_{-1,1}$ to $(-1/2, 1/2)$ is not maximal, even though it solves the IVP there.
4. Sketch $Y_{-1,1}$ and identify precisely which local uniqueness hypothesis fails. Does a maximal interval imply a unique solution?

Question 6 – Solution

Strategy. Check continuation at the joins directly. The length of a solution's interval and uniqueness of the solution are different questions.

Step 1: Verify all three pieces and the joins. For $x < a$,

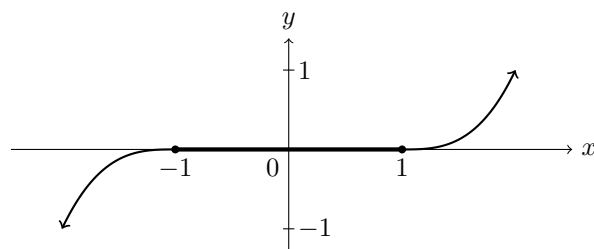
$$Y'_{a,b} = 3(x-a)^2 = 3|(x-a)^3|^{2/3},$$

and the same computation holds with b for $x > b$. On the middle piece both sides vanish.

At $x = a$, the values from both sides are 0, and the one-sided derivatives tend to 0. The derivative at the join itself is 0 by the difference quotient. The same holds at b , including the case $a = b = 0$. Thus $Y_{a,b}$ is continuously differentiable on $\mathbb{R}$, satisfies the equation everywhere, and has $Y_{a,b}(0) = 0$.

Step 2: Exhibit distinct maximal solutions. Two examples are $y(x) = 0$ for all x and $y(x) = x^3 = Y_{0,0}(x)$. Their maximal intervals are both $\boxed{\mathbb{R}}$: no larger real interval exists. The function $Y_{-1,1}$ is a third global solution.

Step 3: Separate a restriction from a maximal solution. On $(-1/2, 1/2)$, $Y_{-1,1} = 0$, but it extends as $Y_{-1,1}$ to the whole real line. Its restriction is therefore not maximal. That restriction can also be extended as the identically zero solution; an extension need not be unique here.



Solid graph: $Y_{-1,1}$; thick segment: $y = 0$

Step 4: Locate the failed uniqueness condition. The right-hand side is continuous, but it is not locally Lipschitz in y at 0:

$$\frac{|3|y|^{2/3} - 0|}{|y - 0|} = 3|y|^{-1/3} \rightarrow \infty.$$

Consequently the usual local uniqueness hypothesis is unavailable at the initial point. The explicit examples prove nonuniqueness, even though every exhibited global solution has the same maximal interval $\mathbb{R}$.