

**Question 5**

Consider

$$y' = \frac{1}{3y^2 - 1}, \quad y(0) = 0,$$

with  $y \neq \pm 1/\sqrt{3}$ .

**Tasks**

1. Derive an implicit equation and identify the portion that defines the IVP's branch as a function of  $x$ .
2. Determine the exact maximal interval containing 0, using monotonicity rather than a cubic-root formula.
3. Find the one-sided limits of  $y$  and  $y'$  at both endpoints and justify maximality.
4. Sketch the selected branch, and explain why other roots of the same cubic cannot be spliced onto it to extend this classical solution.

## Question 5 – Solution

**Strategy.** Regard  $x$  as a function of  $y$ . Its turning points identify where the selected inverse branch stops being a regular graph.

**Step 1: Integrate without solving the cubic.** The chain rule gives

$$\frac{d}{dx}(y^3 - y) = (3y^2 - 1)y' = 1, \quad \boxed{x = y^3 - y}.$$

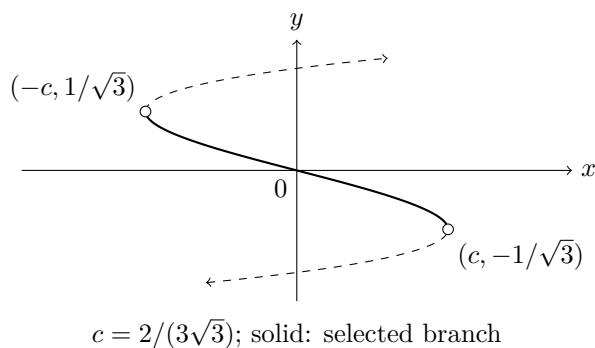
The initial value sets the integration constant to zero. Put  $h(y) = y^3 - y$ . On  $-1/\sqrt{3} < y < 1/\sqrt{3}$ ,  $h'(y) < 0$ , so  $h$  is strictly decreasing and has a differentiable inverse there.

**Step 2: Map the branch endpoints.** Let  $c = 2/(3\sqrt{3})$ . Then  $h(1/\sqrt{3}) = -c$  and  $h(-1/\sqrt{3}) = c$ . Hence the inverse through  $(0, 0)$  exists exactly on

$$\boxed{I = (-c, c) = \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)}.$$

Differentiating its defining relation recovers  $y' = 1/(3y^2 - 1)$  and verifies the original equation on this branch.

**Step 3: Analyze the folds.** As  $x \downarrow -c$ ,  $y \rightarrow 1/\sqrt{3}$ ; as  $x \uparrow c$ ,  $y \rightarrow -1/\sqrt{3}$ . In both cases  $3y^2 - 1 \rightarrow 0^-$ , so  $y' \rightarrow -\infty$ . Both finite limiting values are excluded by the original equation.



**Step 4: Reject a change of root.** The dashed pieces show other parts of the algebraic curve. A continuous switch at a fold must include an excluded value of  $y$  with unbounded slope; a switch to a different root at that  $x$  would be discontinuous. Neither gives a classical extension. The cubic relation alone does not specify the IVP's interval or branch.