

Question 3

Consider

$$y' = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2,$$

with $x \neq 1$ and real y .

Tasks

1. Use $u = e^y$ to solve the IVP, explicitly enforcing the range of this substitution.
2. Determine the maximal interval containing 2; compare its left endpoint with the singular point of the coefficient $1/(x-1)$.
3. Verify the solution and calculate its limiting behavior at both ends of its maximal interval.
4. Explain why neither continuing the linear transformed solution through a zero nor replacing $\ln u$ by $\ln |u|$ produces a continuation of the original solution.

Question 3 – Solution

Strategy. The exponential substitution is invertible only for $u > 0$. Its range can restrict the interval before a coefficient singularity is reached.

Step 1: Transform and invert. On the component $x > 1$ containing 2,

$$u' = e^y y' = \frac{1}{x-1}, \quad u = \ln(x-1) + C.$$

Since $u(2) = 2$, $C = 2$. Therefore

$$y = \ln(2 + \ln(x-1))$$

is a real solution only where $2 + \ln(x-1) > 0$.

Step 2: Find the interval. The positivity condition is $x > 1 + e^{-2}$. Let $\alpha = 1 + e^{-2}$; then

$$I = (\alpha, \infty).$$

In particular, $1 < \alpha < 2$. The solution fails before reaching $x = 1$, even though the original right-hand side is smooth at every finite point (α, y) .

Step 3: Verify and analyze the ends. Writing $u = 2 + \ln(x-1) > 0$ gives

$$y' = \frac{1}{(x-1)u} = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2.$$

As $x \downarrow \alpha$, $u \downarrow 0$ and $y \rightarrow -\infty$, preventing a finite continuous extension. As $x \rightarrow \infty$, $u \rightarrow \infty$ and $y \rightarrow \infty$, but this occurs only at an infinite endpoint and does not shorten the interval.

Step 4: Reject an invalid inverse. The transformed function continues on $(1, \infty)$ but becomes negative on $(1, \alpha)$. No real y can have $e^y = u \leq 0$.

On that negative portion, $z = \ln |u|$ would satisfy $z' = 1/((x-1)u) < 0$, whereas $e^{-z}/(x-1) = 1/((x-1)|u|) > 0$. Thus the absolute-value replacement fails the original equation and cannot repair the lost exponential range.