

Question 10

For a homogeneous equation $y' = F(y/x)$ on $x > 0$, introduce the logarithmic independent variable $t = \ln x$ and the ratio $v(t) = y(e^t)/e^t$.

We want F to be a polynomial of degree at most two such that, in the transformed equation, $v = 1$ and $v = 3$ are equilibrium values and $dv/dt = -1$ when $v = 2$.

Tasks

1. Derive the equation for dv/dt in terms of F and v .
2. Determine the unique polynomial F satisfying the design requirements.
3. Solve the resulting original IVP with $y(1) = 2$, showing the transformed integration and recovery of y .
4. Find the limits of y/x as $x \downarrow 0$ and as $x \rightarrow \infty$. Identify the original solutions represented by the equilibrium ratios and explain why they are not constant functions y . Decide whether the selected original $y(x)$ increases or decreases.

Question 10 – Solution

Strategy. The logarithmic independent variable turns multiplication of x into translation of t and removes the factor x from the ratio equation.

Step 1: Derive and design the transformed equation. Writing $y = xv(\ln x)$ gives $y' = v + dv/dt$, so

$$\frac{dv}{dt} = F(v) - v.$$

This polynomial has roots 1, 3 and degree at most two, hence equals $k(v-1)(v-3)$. Its value -1 at $v = 2$ forces $k = 1$. Therefore

$$\boxed{F(v) = v^2 - 3v + 3}, \quad v' = (v-1)(v-3).$$

Step 2: Solve the selected trajectory. The data give $v(0) = 2$, between the equilibria. Separation yields

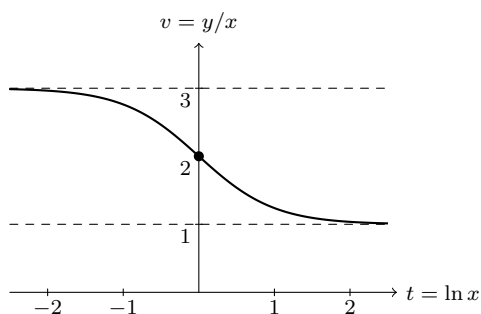
$$\frac{1}{2} \ln \left| \frac{v-3}{v-1} \right| = t + C, \quad C = 0.$$

The ratio is negative on the selected branch, so $(v-3)/(v-1) = -e^{2t}$. Thus

$$v = 1 + \frac{2}{1 + e^{2t}}, \quad \boxed{y = x + \frac{2x}{1 + x^2}, \quad x > 0}.$$

Directly $dv/dt = -4e^{2t}/(1 + e^{2t})^2 = (v-1)(v-3)$, so the chain rule verifies the original equation and $y(1) = 2$.

Step 3: Interpret the limits and equilibria. As $x \downarrow 0$, $t \rightarrow -\infty$ and $y/x \rightarrow 3$; as $x \rightarrow \infty$, $t \rightarrow \infty$ and $y/x \rightarrow 1$. The constant ratios give $\boxed{y = x, y = 3x}$, both valid on $x > 0$. They are straight lines, not constant original functions: constancy applies to the ratio in the new coordinates. Moreover, $y' = F(v) = (v-3/2)^2 + 3/4 > 0$, so the original y increases even though its ratio v decreases.



The transformed ratio decreases from 3 to 1; the dashed levels represent $y = 3x$ and $y = x$.