

Question 9

Consider the equation

$$y' = y^2 + (1 - 2x)y + x^2 - x + 1, \quad y(0) = 2.$$

A particular solution is suggested: $y_p(x) = x$.

Tasks

1. Verify the suggested solution and derive the equation for $z = y - y_p$.
2. On a branch with $z \neq 0$, set $u = 1/z$ and derive a linear equation for u .
3. Solve the IVP and find its maximal interval containing 0.
4. Verify the recovered solution through the substitutions, restore the solution excluded by $u = 1/z$, and explain why this reciprocal is applied to the difference from y_p rather than directly to y .

Question 9 – Solution

Strategy. Subtract a known particular solution so that the constant term cancels, then use a reciprocal on the remaining quadratic equation.

Step 1: Verify and subtract the particular solution. For $y_p = x$, the right-hand side is $x^2 + (1 - 2x)x + x^2 - x + 1 = 1 = y'_p$. Now write $y = x + z$. Substitution and cancellation give

$$1 + z' = 1 + z + z^2, \quad \boxed{z' = z + z^2}, \quad z(0) = 2.$$

Step 2: Make the remaining equation linear. For $z \neq 0$, $u = 1/z$ gives

$$u' = -\frac{z'}{z^2} = -u - 1, \quad u(0) = \frac{1}{2}.$$

Solving $u' + u = -1$ yields $u = Ce^{-x} - 1$, and the initial value requires $C = 3/2$.

Step 3: Recover the IVP and its interval. Thus

$$\boxed{y = x + \frac{1}{\frac{3}{2}e^{-x} - 1}, \quad I = (-\infty, \ln(3/2))}.$$

The denominator is positive before its unique zero at $\ln(3/2)$ and gives $y(0) = 2$. At that endpoint $y \rightarrow +\infty$, preventing a finite differentiable extension.

Step 4: Verify and account for the missing solution. Where $u \neq 0$, $z = 1/u$ satisfies $z' = -u'/u^2 = z + z^2$. Thus $y' = 1 + z + z^2$, which is exactly the original right side after $y = x + z$. The excluded case $z \equiv 0$ restores $\boxed{y = x}$, a global solution with a different initial value.

The original equation contains a nonzero term independent of y . A direct reciprocal $w = 1/y$ would retain a quadratic term in w and would not be linear. Subtracting the known solution first cancels that obstruction. This two-step substitution is the useful structure, not the reciprocal alone.