

Question 8

On $x > 0$, consider

$$x^2 y' + y = e^{-1/x}.$$

We seek a solution that remains bounded as $x \downarrow 0$.

Tasks

1. Change the independent variable to $t = 1/x$ and define $Y(t) = y(1/t)$. Derive the transformed equation carefully using the chain rule.
2. Solve the transformed equation and recover the full original solution family on $x > 0$.
3. Select the unique solution bounded as $x \downarrow 0$. Find its limit at 0, its limit as $x \rightarrow \infty$, and its value at $x = 1$.
4. Verify it in the original equation and explain how the substitution changes the endpoint at which the boundedness condition is imposed.

Question 8 – Solution

Strategy. A reciprocal change of the independent variable absorbs x^2 , while converting a finite endpoint into an infinite one.

Step 1: Transform the derivative. Since $t = 1/x$, $dt/dx = -1/x^2$. Writing $y(x) = Y(t)$ gives

$$x^2 y' = -Y'(t), \quad -Y' + Y = e^{-t}.$$

Thus $\boxed{Y' - Y = -e^{-t}}$ for $t > 0$. The minus sign comes from the reversal of the independent variable.

Step 2: Solve and return to x . The integrating factor is e^{-t} :

$$(e^{-t}Y)' = -e^{-2t}, \quad Y = \frac{1}{2}e^{-t} + Ce^t.$$

Therefore all solutions on the stated half-line are

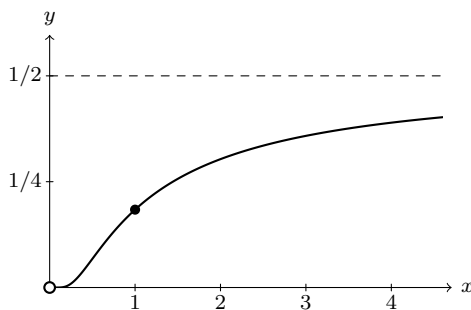
$$\boxed{y(x) = \frac{1}{2}e^{-1/x} + Ce^{1/x}}.$$

Step 3: Apply the endpoint condition. As $x \downarrow 0$, $t \rightarrow +\infty$. The first term tends to zero, whereas any nonzero C makes the second term unbounded. Consequently the unique bounded choice is

$$\boxed{y_b(x) = \frac{1}{2}e^{-1/x}}.$$

Its limits are 0 as $x \downarrow 0$ and $1/2$ as $x \rightarrow \infty$, and $\boxed{y_b(1) = 1/(2e)}$.

Step 4: Verify and interpret the reversal. Differentiation gives $y'_b = e^{-1/x}/(2x^2)$, so $x^2 y'_b + y_b = e^{-1/x}$. The condition near $x = 0$ becomes a condition at $t = \infty$, not at $t = 0$. Conversely, $x \rightarrow \infty$ corresponds to $t \downarrow 0$. The limit at $x = 0$ does not make that point part of the stated domain of the equation.



The bounded solution approaches the excluded point $(0, 0)$ and tends to $1/2$ as $x \rightarrow \infty$.