

Question 7

Consider

$$y' = xe^{-y} - 1, \quad y(0) = 0.$$

Tasks

1. Set $u = e^y$ and derive a linear differential equation for u .
2. Solve it using the initial condition and recover y .
3. Prove that the selected transformed solution stays positive for every real x . Find the minimum value of y and where it occurs.
4. The same linear equation has the solution $u = x - 1$. Determine precisely where it produces a real original solution, and explain why a valid transformed solution need not be invertible everywhere.

Question 7 – Solution

Strategy. Exponentiating the unknown makes the reciprocal exponential linear, but the inverse logarithm requires a positive transformed value.

Step 1: Transform and solve. For $u = e^y$, $u' = e^y y' = x - u$. The initial value is $u(0) = 1$, so

$$u' + u = x, \quad (e^x u)' = x e^x,$$

which gives $u = x - 1 + C e^{-x}$. The data select $C = 2$, hence

$$y(x) = \ln(x - 1 + 2e^{-x}).$$

Step 2: Establish the inverse domain and minimum. Let $U(x) = x - 1 + 2e^{-x}$. Then $U' = 1 - 2e^{-x}$ and $U'' = 2e^{-x} > 0$. Its unique global minimum occurs at $x = \ln 2$ and equals $\ln 2 > 0$. Therefore the logarithm is defined and smooth for all real x , so the IVP interval is $\boxed{\mathbb{R}}$.

Since the logarithm is increasing, the solution has unique minimum

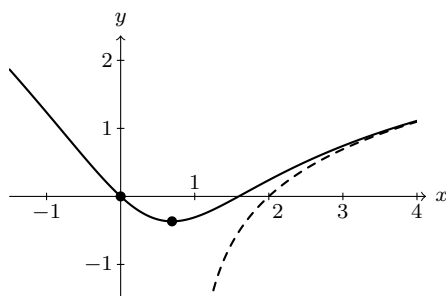
$$y(\ln 2) = \ln(\ln 2).$$

Verification is immediate from $U' = x - U$: $y' = U'/U = x/U - 1 = x e^{-y} - 1$, and $y(0) = \ln 1 = 0$.

Step 3: Test another transformed solution. The linear solution $u = x - 1$ is defined everywhere, but $u > 0$ only for $x > 1$. It gives

$$y = \ln(x - 1), \quad x > 1.$$

On that interval, $y' = 1/(x - 1) = x/(x - 1) - 1$, verifying the original equation. At $x \downarrow 1$, $y \rightarrow -\infty$, and for $x \leq 1$ no real logarithm of u exists. The map $y \mapsto e^y$ has range $(0, \infty)$; solving the transformed linear equation does not remove that range restriction.



Solid: the global IVP solution. Dashed: $\ln(x - 1)$, which is valid only for $x > 1$.