

Question 6

Consider an equation with parallel affine expressions:

$$y' = \frac{2x + y + 1}{4x + 2y + 3}, \quad y(0) = 0,$$

where $4x + 2y + 3 \neq 0$.

Tasks

1. Explain why no translation of x, y removes the constants in both numerator and denominator simultaneously.
2. Use $u = 2x + y + 1$ to derive a separable equation, then obtain an implicit formula for the IVP.
3. Find the solution corresponding to a constant u that would be lost during separation, and verify it in the original equation.
4. Prove that the selected implicit formula defines the IVP solution for every real x , while remaining inside the original coefficient domain.

Question 6 – Solution

Strategy. When the affine forms cannot be centered together, use their shared linear combination directly.

Step 1: Reject a simultaneous centering. A translation by (h, k) would require $2h + k + 1 = 0$ and $4h + 2k + 3 = 0$. Doubling the first equation gives $4h + 2k + 2 = 0$, contradicting the second. Their zero lines are parallel and distinct.

Step 2: Reduce and integrate. Set $u = 2x + y + 1$. Then the denominator is $2u + 1$, and

$$u' = 2 + y' = \frac{5u + 2}{2u + 1}, \quad \frac{2u + 1}{5u + 2} du = dx.$$

The original domain excludes $u = -1/2$. For $u \neq -2/5$, an antiderivative is

$$H(u) = \frac{2}{5}u + \frac{1}{25} \ln |5u + 2|.$$

Since $u(0) = 1$, the IVP relation is

$$H(u) = x + \frac{2}{5} + \frac{1}{25} \ln 7, \quad y = u - 2x - 1.$$

Step 3: Restore the lost constant combination. The transformed equation admits $u \equiv -2/5$, giving $y = -2x - 7/5$. Its numerator is $-2/5$ and its denominator is $1/5$, so the right side is $-2 = y'$. It is a global solution, but its initial value is not 0.

Step 4: Prove the selected branch is global. The initial value lies in $u > -2/5$. On this interval,

$$H'(u) = \frac{2u + 1}{5u + 2} > 0,$$

and $H(u)$ tends to $-\infty$ at the left endpoint and to $+\infty$ as $u \rightarrow \infty$. Thus the implicit relation gives exactly one differentiable u for every real x . Also $2u + 1 > 1/5$, so the original denominator never vanishes. Differentiating $H(u) = x + H(1)$ gives the transformed equation and then the original equation; this verifies the unique global IVP branch.