

Question 4

Consider the nonseparable-looking equation

$$y' = (x + y)^2 - 1, \quad y(0) = 1.$$

Tasks

1. Set $u = x + y$. Derive its differential equation, including the derivative of x .
2. Solve the transformed IVP, recover y , and find its maximal interval containing 0.
3. Determine the minimum of the selected solution and verify the original equation.
4. Find the original solution represented by the constant transformed value $u = 0$. Explain why dividing by u^2 would miss it and why it is not a constant function y .

Question 4 – Solution

Strategy. Replace the repeated linear combination by one variable. The extra derivative of x cancels the constant term in the equation.

Step 1: Reduce to a separable IVP. For $u = x + y$, $u' = 1 + y' = u^2$, and $u(0) = 1$. Separating on the nonzero branch gives

$$-\frac{1}{u} = x + C, \quad C = -1, \quad u = \frac{1}{1-x}.$$

Thus

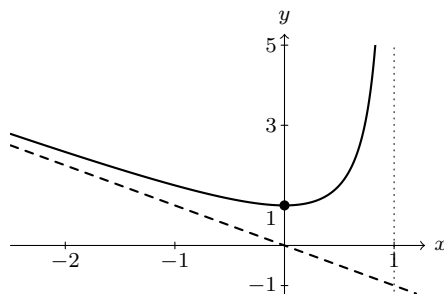
$$y = -x + \frac{1}{1-x}, \quad I = (-\infty, 1).$$

The only finite obstruction is the pole at $x = 1$, where $y \rightarrow +\infty$ from the left.

Step 2: Find the minimum and verify. Differentiation gives $y' = -1 + (1-x)^{-2}$. On I this is negative for $x < 0$, zero at 0, and positive for $0 < x < 1$. Hence the unique global minimum on I is $y(0) = 1$.

Also $x + y = 1/(1-x)$, so $(x + y)^2 - 1 = (1-x)^{-2} - 1 = y'$, and the initial value checks directly.

Step 3: Restore the constant transformed branch. The equation $u' = u^2$ also has $u \equiv 0$. In the original variables it gives $y = -x$ on all of $\mathbb{R}$. Its derivative is -1 and its right-hand side is $0^2 - 1 = -1$, so it is valid. Division by u^2 excludes it. A constant value of a combination of x and y need not represent a constant value of y .



Solid: IVP solution, with minimum $(0, 1)$. Dashed: the restored solution $y = -x$.