

Question 3

Consider an IVP on the negative half-line:

$$y' = \frac{y}{x} + \frac{x}{y}, \quad y(-1) = 1, \quad x < 0,$$

with the original equation defined only when $y \neq 0$.

Tasks

1. Use $v = y/x$ and separate the equation, keeping the correct logarithmic absolute value.
2. Select both the constant and the sign of the transformed solution from the initial condition.
3. Recover y and determine its maximal interval containing -1 within $x < 0$.
4. Verify the equation and describe the finite endpoint. Explain why replacing $\ln|x|$ by $\ln x$, or choosing the positive sign for v , would fail here.

Question 3 – Solution

Strategy. The substitution is valid for negative x too, but its initial ratio is negative and the logarithm must respect the domain.

Step 1: Transform and integrate. From $y = xv$ and $y' = v + xv'$,

$$xv' = \frac{1}{v}, \quad v \, dv = \frac{dx}{x}, \quad v^2 = 2 \ln |x| + C.$$

At $x = -1$, $v = y/x = -1$, so $C = 1$. Continuity and the nonzero ratio select

$$v = -\sqrt{1 + 2 \ln |x|}.$$

Step 2: Recover the branch and domain. The radicand must be strictly positive, since zero would give the excluded value $y = 0$. Thus $|x| > e^{-1/2}$. On the negative half-line this gives

$$\boxed{y = -x\sqrt{1 + 2 \ln |x|}, \quad I = (-\infty, -e^{-1/2})}.$$

It is positive and gives $y(-1) = 1$.

Step 3: Verify and check the endpoint. Let $D = 1 + 2 \ln |x|$. Since $D' = 2/x$,

$$y' = -\sqrt{D} - \frac{1}{\sqrt{D}}, \quad \frac{y}{x} + \frac{x}{y} = -\sqrt{D} - \frac{1}{\sqrt{D}}.$$

At the finite endpoint approached from the left, $D \downarrow 0$, so $y \rightarrow 0$ from above and $y' \rightarrow -\infty$. The original equation is undefined there, and no differentiable extension through it is possible.

Step 4: Identify the two domain mistakes. The real logarithm $\ln x$ is not defined for these negative x ; the antiderivative is $\ln |x|$. Also $v > 0$ would give $y = xv < 0$ on this half-line, contradicting $y(-1) = 1$. The sign must be selected for the transformed ratio, not guessed from the sign of y alone.