

Question 1

Consider the homogeneous first-order equation

$$y' = \frac{y}{x} + \sqrt{1 + (y/x)^2}, \quad y(1) = 0, \quad x > 0.$$

Here “homogeneous” means that the right-hand side depends only on y/x ; it does not mean a homogeneous linear equation.

Tasks

1. Set $v = y/x$ and derive the equation for v , showing the product-rule term in y' .
2. Solve the transformed IVP and recover an explicit formula for y .
3. Verify the original equation, taking care with the nonnegative square root, and find the maximal interval within $x > 0$.
4. A student uses $y' = xv'$ when $y = xv$. Identify the omitted term and explain why that error changes the transformed equation.

Question 1 – Solution

Strategy. The ratio substitution removes the repeated y/x , but differentiating $y = xv(x)$ requires both product-rule terms.

Step 1: Transform the equation. Since $y' = v + xv'$, substitution gives

$$v + xv' = v + \sqrt{1 + v^2}, \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

The initial value is $v(1) = 0$. An antiderivative on the left is $\ln(v + \sqrt{1 + v^2})$, whose logarithm has a strictly positive argument for every real v . Hence

$$\ln(v + \sqrt{1 + v^2}) = \ln x.$$

Step 2: Recover the original unknown. Exponentiating gives $v + \sqrt{1 + v^2} = x$. Squaring $\sqrt{1 + v^2} = x - v$ and simplifying gives

$$v = \frac{x^2 - 1}{2x}, \quad \boxed{y = \frac{x^2 - 1}{2}}.$$

This step is valid: for the recovered expression, $x - v = (x^2 + 1)/(2x) > 0$ on $x > 0$, so the unsquared relation also holds.

Step 3: Verify the branch and interval. For this y ,

$$\sqrt{1 + (y/x)^2} = \frac{x^2 + 1}{2x}, \quad \frac{y}{x} + \sqrt{1 + (y/x)^2} = x = y'.$$

Also $y(1) = 0$. The formula satisfies the original equation for every $x > 0$, so its maximal interval there is $\boxed{(0, \infty)}$. The polynomial has a value at 0, but the given equation does not; an algebraic extension alone does not extend this solution through the excluded point.

Step 4: Diagnose the missing derivative term. The omitted term is v . Writing only xv' would produce $xv' = v + \sqrt{1 + v^2}$ instead of $xv' = \sqrt{1 + v^2}$. The two equations describe different ratio functions, so the product rule is essential to the reduction.