

Question 10

Consider positive solutions, near $x = 0$, of the exponent-dependent IVP

$$y' + y = y^n, \quad y(0) = 2, \quad n \in \mathbb{R}.$$

For $y > 0$, interpret $y^n = \exp(n \ln y)$.

Tasks

1. Solve the cases $n = 0$ and $n = 1$ directly. Explain why they do not require the usual nonlinear Bernoulli reduction.
2. For $n \neq 1$, derive the solution formula and state the positivity condition needed to interpret its real power.
3. Check that the general formula agrees with the direct answer at $n = 0$ and verifies the initial condition for every admissible n .
4. For each fixed real x , prove that the formula tends to the $n = 1$ solution as $n \rightarrow 1$. Do not substitute $n = 1$ into an expression with exponent $1/(1 - n)$.

Question 10 – Solution

Strategy. Handle the exceptional equations directly, then analyze the apparent singularity of the general expression through its logarithm.

Step 1: Solve the elementary exceptional cases. If $n = 0$, the equation is $y' + y = 1$, giving $y = 1 + e^{-x}$. If $n = 1$, the two y terms cancel, giving $y = 2$. Both are positive on $\mathbb{R}$; the first is linear and the second has zero derivative.

Step 2: Derive the parameterized formula. For $n \neq 1$, set $h = 1 - n \neq 0$ and $v = y^h$. Then

$$v' + hv = h, \quad v(0) = 2^h,$$

so $v = 1 + (2^h - 1)e^{-hx}$. Therefore

$$y_n(x) = \left[1 + (2^{1-n} - 1)e^{-(1-n)x} \right]^{1/(1-n)}.$$

Use the connected interval through 0 on which the bracket is strictly positive. This is necessary because the original positive solution has $v = y^h > 0$; a negative base cannot be accepted by choosing a special rational exponent.

Step 3: Check the consistency conditions. At $n = 0$, the formula becomes $1 + e^{-x}$. At $x = 0$, its bracket is 2^{1-n} , and the positive-real power gives 2. Conversely, differentiating $v = y^h$ on the stated interval recovers the original equation, so the inversion is valid there.

Step 4: Take the limit without dividing by zero. For fixed x , let $B(h) = 1 + (e^{h \ln 2} - 1)e^{-hx}$. Since $B(0) = 1$, it is positive for all sufficiently small $|h|$. Now

$$\ln y_n = \frac{\ln B(h)}{h}, \quad B'(0) = \ln 2.$$

The derivative definition gives $\lim_{h \rightarrow 0} \ln B(h)/h = \ln 2$. Exponentiating yields $\lim_{n \rightarrow 1} y_n(x) = 2$, exactly the direct $n = 1$ solution. Joint continuity on the compact segment between 0 and the fixed x keeps the bracket positive there for small $|h|$, so these values belong to the IVP branch.