

Question 9

Seek strictly positive, 2π -periodic solutions, defined for every real x , of

$$y' + y = (2 + \sin x)y^2.$$

Tasks

1. Use the reciprocal substitution to find the general transformed solution.
2. Prove that there is exactly one strictly positive 2π -periodic solution and find its initial value at $x = 0$.
3. Determine its minimum and maximum over one period.
4. Show that changing the initial value to $y(0) = 1/2$ causes a finite positive-time blow-up, even though the periodic solution remains bounded. Verify the original equation for the periodic solution.

Question 9 – Solution

Strategy. Periodicity removes the exponentially growing homogeneous term in the reciprocal variable; positivity must then be checked before inversion.

Step 1: Solve the linear equation. With $v = 1/y$, the equation becomes $v' - v = -(2 + \sin x)$. A particular solution and the general solution are

$$v_p = 2 + \frac{\sin x + \cos x}{2}, \quad v = v_p + Ce^x.$$

Indeed $v'_p - v_p = -2 - \sin x$.

Step 2: Select and bound the periodic solution. If positive y is 2π -periodic, then so is $v = 1/y$. The term Ce^x is periodic only if $C = 0$. Since

$$2 - \frac{\sqrt{2}}{2} \leq v_p \leq 2 + \frac{\sqrt{2}}{2},$$

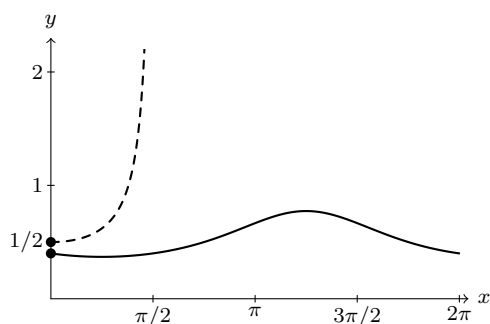
its reciprocal exists globally and is positive. Therefore

$$y_p = \frac{1}{2 + (\sin x + \cos x)/2}, \quad y_p(0) = \frac{2}{5}.$$

Its minimum is $1/(2 + \sqrt{2}/2)$ at $x = \pi/4$, and its maximum is $1/(2 - \sqrt{2}/2)$ at $x = 5\pi/4$, modulo 2π .

Step 3: Check the perturbed initial condition. For $y(0) = 1/2$, $v(0) = 2$, so $C = -1/2$. The function $v = v_p - e^x/2$ starts positive, but $v(2\pi) = 5/2 - e^{2\pi}/2 < 0$. It therefore has a first zero $T \in (0, 2\pi)$. At this zero, $v'(T) = -(2 + \sin T) < 0$. Hence $v \downarrow 0$ from the positive side and $y = 1/v \rightarrow +\infty$ as $x \uparrow T$.

Step 4: Verify the periodic branch. Since $v'_p = v_p - (2 + \sin x)$, differentiation gives $y'_p = -y_p + (2 + \sin x)y_p^2$, as required.



Solid: the positive periodic solution. Dashed: $y(0) = 1/2$, which blows up before a full cycle.