

Question 8

Consider

$$y' + xy = e^{x^2} y^2, \quad y(0) = a, \quad a \in \mathbb{R}.$$

Define $J(x) = \int_0^x e^{t^2/2} dt$; an elementary antiderivative is not required.

Tasks

1. Solve the IVP for $a \neq 0$ by the Bernoulli substitution, expressing the answer in terms of J . Treat $a = 0$ separately.
2. Prove that J is strictly increasing onto $\mathbb{R}$ and that the equation $J(T) = 1/a$ has exactly one real solution.
3. Determine the maximal interval containing 0 for each sign of a , and the sign of the divergence at its finite endpoint.
4. Verify the formula using the fundamental theorem of calculus and explain why an exact definite-integral expression is a complete answer.

Question 8 – Solution

Strategy. Keep the forcing integral exact, then use its monotonicity to determine the entire admissible denominator interval.

Step 1: Linearize and integrate. For $a \neq 0$, set $v = 1/y$. Then $v' - xv = -e^{x^2}$, with integrating factor $e^{-x^2/2}$. Thus

$$(e^{-x^2/2}v)' = -e^{x^2/2}, \quad v = e^{x^2/2} \left(\frac{1}{a} - J(x) \right).$$

Inverting gives

$$y(x) = \frac{ae^{-x^2/2}}{1 - aJ(x)}.$$

For $a = 0$, $y \equiv 0$ is the unique solution, since the original right-hand side is smooth in y .

Step 2: Locate the unique obstruction. By the fundamental theorem, $J' = e^{x^2/2} > 0$. The integrand is even, so J is odd. Also $J(x) \geq x$ for $x \geq 0$, so its limits at the two infinities are $\pm\infty$. Therefore it is one-to-one and onto, and there is exactly one T with $J(T) = 1/a$.

Step 3: Select the interval and sign. If $a > 0$, then $T > 0$, and the maximal interval is $(-\infty, T)$. The denominator approaches zero from above as $x \uparrow T$, so $y \rightarrow +\infty$. If $a < 0$, then $T < 0$, and the maximal interval is (T, ∞) ; as $x \downarrow T$, the denominator approaches zero from above and $y \rightarrow -\infty$. A pole cannot be crossed by a finite differentiable continuation.

Step 4: Verify and interpret. With $D = 1 - aJ$, differentiation using $J' = e^{x^2/2}$ gives

$$y' = -xy + \frac{a^2}{D^2} = -xy + e^{x^2} y^2.$$

Also $J(0) = 0$ gives $y(0) = a$. The definite integral specifies an exact differentiable function and uniquely determines the endpoint; elementary notation is unnecessary for either the solution or its interval.