

**Question 7**

Suppose  $y_1, y_2$  are positive solutions on an interval of

$$y' + p(x)y = q(x)y^3.$$

Let  $0 < \alpha < 1$ .

**Tasks**

1. Prove that  $v = \alpha y_1^{-2} + (1 - \alpha)y_2^{-2}$  solves the transformed linear equation and gives a positive solution  $y = v^{-1/2}$ .
2. Show that this new solution lies between  $y_1$  and  $y_2$  pointwise.
3. Apply the construction with  $\alpha = 1/4$  to the equation  $y' + xy = xy^3$  and the solutions  $y_1 = 1$ ,  $y_2 = (1 + e^{x^2})^{-1/2}$ . Verify the supplied solutions through their transforms.
4. Prove that the ordinary arithmetic mean  $m = (y_1 + y_2)/2$  of these two supplied solutions fails the nonlinear equation at  $x = 1$ .

## Question 7 – Solution

**Strategy.** Affine combinations preserve the common forcing in the transformed linear equation; they do not generally preserve the original cubic equation.

**Step 1: Combine the transformed solutions.** Set  $v_i = y_i^{-2}$ . Both satisfy  $v'_i - 2pv_i = -2q$ . Multiplying by  $\alpha$  and  $1 - \alpha$  and adding yields  $v' - 2pv = -2q$ . Positivity of  $v_1, v_2$  implies  $v > 0$ , so  $y = v^{-1/2}$  is well defined. Differentiating this inverse recovers  $y' + py = qy^3$ .

**Step 2: Prove the pointwise bound.** At each point,  $v$  lies between  $v_1$  and  $v_2$ . The function  $u \mapsto u^{-1/2}$  is decreasing on  $u > 0$ , so reversing their order still puts  $y$  between  $y_1$  and  $y_2$ . Equality occurs when the supplied values agree; otherwise the bounds are strict.

**Step 3: Apply the construction.** For the given equation, the transformed equation is  $v' - 2xv = -2x$ . The supplied transforms are  $v_1 = 1$  and  $v_2 = 1 + e^{x^2}$ ; direct differentiation verifies both. Their weighted combination gives

$$v = 1 + \frac{3}{4}e^{x^2}, \quad \boxed{y = \frac{1}{\sqrt{1 + \frac{3}{4}e^{x^2}}}} \quad (x \in \mathbb{R}).$$

It is positive and lies strictly between the two supplied solutions.

**Step 4: Reject ordinary superposition.** Because the original functions solve the equation,

$$m' + xm = \frac{x}{2}(y_1^3 + y_2^3).$$

At  $x = 1$ , put  $a = y_1(1)$  and  $b = y_2(1)$ . They are distinct and positive. The residual is

$$m' + m - m^3 = \frac{a^3 + b^3}{2} - \frac{(a + b)^3}{8} = \frac{3}{8}(a + b)(a - b)^2 > 0.$$

Thus the arithmetic mean fails the nonlinear equation. The valid averaging occurs after the Bernoulli transformation.