

Question 6

On $x > 0$, two positive solutions of one unknown Bernoulli equation

$$y' + p(x)y = q(x)y^2$$

are known exactly:

$$y_1(x) = \frac{1}{1+x}, \quad y_2(x) = \frac{1}{1+2x}.$$

Assume p and q are continuous there.

Tasks

1. Transform both solutions using $v = 1/y$ and recover $p(x)$ and $q(x)$ uniquely.
2. Verify both supplied solutions directly in the recovered nonlinear equation.
3. Solve that equation with the different condition $y(1) = 2$, and find its maximal interval within $x > 0$.
4. Explain why subtracting the original nonlinear solutions is less useful here than subtracting their transformed equations, and identify the solution lost by taking reciprocals.

Question 6 – Solution

Strategy. The reciprocal transformation turns coefficient recovery into two linear identities; their difference eliminates the forcing term.

Step 1: Recover the coefficients. Each transformed solution satisfies $v' - pv = -q$. Here $v_1 = 1 + x$ and $v_2 = 1 + 2x$, so

$$1 - p(1 + x) = -q, \quad 2 - p(1 + 2x) = -q.$$

Subtracting gives $1 - px = 0$. Since $x > 0$, this fixes $p = 1/x$ uniquely. Substituting back gives $q = 1/x$. Thus

$$\boxed{y' + \frac{y}{x} = \frac{y^2}{x}}.$$

Step 2: Verify the given data. For $y = 1/(1 + kx)$, with $k = 1$ or 2 ,

$$y' + \frac{y}{x} = -\frac{k}{(1 + kx)^2} + \frac{1}{x(1 + kx)} = \frac{1}{x(1 + kx)^2} = \frac{y^2}{x}.$$

Both functions are positive and defined throughout $x > 0$.

Step 3: Construct the new solution. The transformed equation is $v' - v/x = -1/x$. Multiplying by $1/x$ gives $(v/x)' = -1/x^2$, hence $v = 1 + Cx$. Since $y(1) = 2$, $v(1) = 1/2$, so $C = -1/2$. Therefore

$$\boxed{y = \frac{1}{1 - x/2}, \quad I = (0, 2)}.$$

This is the maximal interval within the specified domain through 1: the solution has a pole at 2, and the coefficients are undefined at 0.

Step 4: Explain the advantage and restore zero. Subtracting the transformed equations gives the homogeneous linear relation $\Delta v' - p\Delta v = 0$. Subtracting the original equations leaves $q(y_2^2 - y_1^2)$ and does not give that linear elimination. The function $\boxed{y \equiv 0}$ satisfies the recovered equation but is excluded by the reciprocal transformation.