

Question 5

For each real initial value a , consider forward solutions of

$$y' + y = y^3, \quad y(0) = a, \quad t \geq 0,$$

where the prime denotes d/dt .

Tasks

1. Solve using $v = y^{-2}$ when $a \neq 0$, retaining the sign of a . Treat $a = 0$ separately.
2. Classify all a according to whether the forward solution tends to zero, is constant, or blows up in finite time.
3. In the blow-up cases, find the exact lifetime and the sign of the divergence.
4. Verify the solution and explain why the threshold initial values are not included in either neighboring regime.

Question 5 – Solution

Strategy. A linear denominator exposes the threshold, while the sign of the original solution must be carried separately.

Step 1: Solve the transformed IVP. For $a \neq 0$, $v = y^{-2}$ gives $v' - 2v = -2$, so

$$v = 1 + (a^{-2} - 1)e^{2t}, \quad y = \frac{a}{\sqrt{a^2 + (1 - a^2)e^{2t}}}.$$

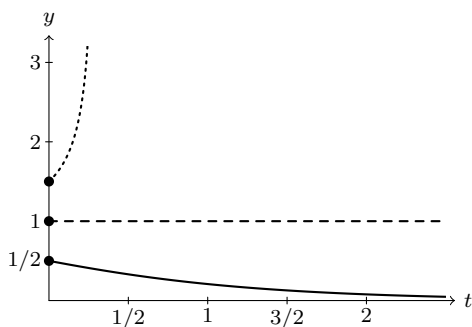
The denominator is the positive square root on the interval through 0. For $a = 0$, the unique solution is $y \equiv 0$; the smooth right-hand side $y^3 - y$ ensures local uniqueness.

Step 2: Classify the thresholds. If $|a| < 1$, the denominator is positive for every $t \geq 0$ and $y \rightarrow 0$. This includes $a = 0$. If $a = \pm 1$, the formula is constant: $y \equiv \pm 1$. If $|a| > 1$, the denominator first vanishes at

$$T = \frac{1}{2} \ln \frac{a^2}{a^2 - 1}.$$

The maximal forward lifetime is $0 \leq t < T$. As $t \uparrow T$, $y \rightarrow +\infty$ for $a > 1$ and $y \rightarrow -\infty$ for $a < -1$.

Step 3: Verify and interpret. For nonzero branches, $v' = -2y^{-3}y' = 2v - 2$ implies $y' = -y + y^3$, and the formula gives $y(0) = a$. The three constants 0, 1, -1 also check directly. At $a = \pm 1$ the exponential term in v is exactly zero; these solutions neither decay nor blow up. A threshold case must be substituted into the equation, not inferred from neighboring values.



Solid: $a = 1/2$. Dashed: $a = 1$. Dotted: $a = 3/2$. Negative solutions are reflections.