

Question 3

Seek nonnegative, continuously differentiable solutions for $t \geq 0$ of

$$y' + 2y = 2\sqrt{y}, \quad y(0) = 0,$$

where the prime denotes d/dt and $\sqrt{y}$ is the nonnegative square root.

Tasks

1. On an interval where $y > 0$, derive and solve the equation for $w = \sqrt{y}$.
2. Find every solution that waits at zero until a time $b \geq 0$ and then becomes positive. Include the possibility of staying zero forever.
3. Check differentiability and the original equation at the joining time, and justify that your list is exhaustive for $t \geq 0$.
4. Explain why squaring a linear formula with $w < 0$ does not give a valid solution of the original equation on that interval.

Question 3 – Solution

Strategy. The linearized equation applies only on positive pieces; restore zero intervals and enforce the sign of the square root.

Step 1: Solve a positive piece. Writing $y = w^2$ with $w > 0$ gives $2ww' + 2w^2 = 2w$, hence $w' + w = 1$. Thus $w = 1 + Ce^{-t}$. If positivity starts at b with $w(b) = 0$, then $w = 1 - e^{-(t-b)}$, positive precisely for $t > b$.

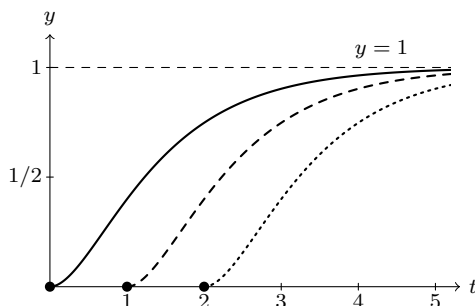
Step 2: Restore the zero intervals. For each finite $b \geq 0$, define

$$y_b(t) = \begin{cases} 0, & 0 \leq t \leq b, \\ (1 - e^{-(t-b)})^2, & t \geq b. \end{cases}$$

Also include $y \equiv 0$, denoted by $b = \infty$. On the positive piece let $r = e^{-(t-b)}$; then $y' = 2r(1 - r)$ and $y' + 2y = 2(1 - r) = 2\sqrt{y}$. At a positive joining time, both one-sided derivatives are 0; if $b = 0$, the right derivative is 0. Thus the solution is C^1 on its stated domain and satisfies the equation at the join.

Step 3: Prove completeness and reject a false extension. Any positive component must start from zero at its finite left endpoint, including $b = 0$, and the linear formula above fixes that component uniquely. It stays positive forever after b , so it cannot end at another zero. Hence there is at most one positive component and the list is exhaustive.

For $t < b$, the same linear expression has $w < 0$. If it is squared, $\sqrt{w^2} = |w| = -w$, while the computed left side is $2w$. Thus it fails the original equation there. Squaring cannot remove this sign restriction.



Solid: $b = 0$. Dashed: $b = 1$. Dotted: $b = 2$. Each is zero before its departure.