

Question 1

A Bernoulli equation has the form $y' + p(x)y = q(x)y^n$. Consider the positive initial-value problem

$$y' + \frac{2}{x}y = x^3y^2, \quad y(1) = 1, \quad x > 0.$$

Tasks

1. For positive y and $n \neq 1$, derive the linear equation satisfied by $v = y^{1-n}$ using the chain rule.
2. Apply that reduction to the given IVP and solve the resulting linear equation with an integrating factor.
3. Find the maximal open interval containing 1, and determine the minimum of the selected solution on that interval.
4. Verify the answer in the original equation and identify a solution omitted by the substitution.

Question 1 – Solution

Strategy. Derive the reduction first, then retain only the interval where its inverse gives the selected positive solution.

Step 1: Reduce to a linear equation. Multiplying by y^{-n} and using $v' = (1 - n)y^{-n}y'$ gives

$$v' + (1 - n)pv = (1 - n)q.$$

Here $n = 2$, so $v = 1/y$ obeys $v' - 2v/x = -x^3$. The integrating factor is x^{-2} , hence

$$(x^{-2}v)' = -x, \quad v = Cx^2 - \frac{x^4}{2}.$$

Since $v(1) = 1$, $C = 3/2$.

Step 2: Invert and analyze. Thus

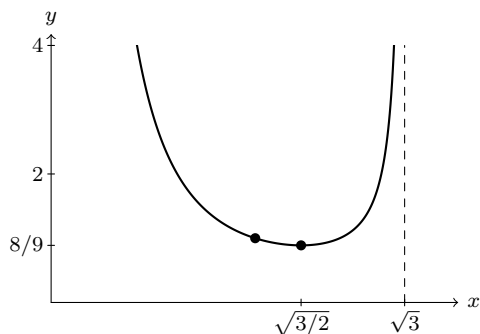
$$y = \frac{2}{x^2(3 - x^2)}, \quad I = (0, \sqrt{3}).$$

The denominator $3x^2 - x^4$ is positive on I , vanishes at its endpoints, and has its unique maximum $9/4$ at $x = \sqrt{3/2}$. The solution therefore has minimum $8/9$ there and tends to $+\infty$ at both ends of I .

Step 3: Verify and restore the lost solution. With $D = 3x^2 - x^4$, $y' = -2(6x - 4x^3)/D^2$, so

$$y' + 2y/x = 4x^3/D^2 = x^3y^2, \quad y(1) = 1.$$

Also $y \equiv 0$ satisfies the original equation on $x > 0$ but is excluded by $v = 1/y$; it does not satisfy this IVP.



The selected branch has minimum $8/9$ and diverges at both interval endpoints.