

Question 9

Consider

$$e^x dx + (1 + y^2) dy = 0, \quad y(0) = 0.$$

An approximate calculation claims that the selected solution passes through $(\ln 2, -1)$.

Tasks

1. Find the exact implicit solution and prove that it defines a unique differentiable graph for all real x .
2. Test the claimed point using the potential, without explicitly solving a cubic.
3. Let y_* be the true value at $x = \ln 2$. Prove $-1 < y_* < 0$ and decide whether the claimed value is too high or too low.
4. Use the mean value theorem to prove the quantitative error bound

$$\frac{1}{6} \leq |y_* - (-1)| \leq \frac{1}{3}.$$

Explain why a potential residual can provide more information than simply declaring an approximate point incorrect.

Question 9 – Solution

Strategy. Use the conserved potential to test the point, then convert its residual into an error bound using a derivative estimate.

Step 1: Find the exact relation. The cross partials are both zero. A potential is $F = e^x + y + y^3/3$, and $F(0, 0) = 1$, so

$$e^x + y + \frac{y^3}{3} = 1.$$

The function $H(u) = u + u^3/3$ is strictly increasing, onto $\mathbb{R}$, and has $H'(u) = 1 + u^2 > 0$. Thus $H(y) = 1 - e^x$ gives exactly one differentiable value for every x . Differentiation recovers the original graph equation.

Step 2: Compute the residual and locate the true value. At the claimed point, $F = 2 - 1 - 1/3 = 2/3$, so the potential residual $F - 1$ is $-1/3$. The point is not on the selected level. At $x = \ln 2$, the true value satisfies $H(y_*) = -1$. Since $H(-1) = -4/3 < -1 < H(0) = 0$, strict monotonicity gives $-1 < y_* < 0$. The claimed value -1 is too low.

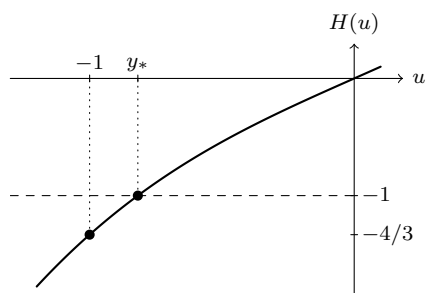
Step 3: Bound the actual error. The mean value theorem gives, for some $\xi \in (-1, y_*)$,

$$\frac{1}{3} = H(y_*) - H(-1) = H'(\xi)(y_* + 1).$$

On $[-1, 0]$, $1 \leq H' \leq 2$, so

$$\frac{1}{6} \leq y_* + 1 = |y_* - (-1)| \leq \frac{1}{3}.$$

The residual, together with monotonicity and derivative bounds, certifies both the direction and the size range of the error; no cubic formula is needed.



The true value y_* lies to the right of -1 because H is strictly increasing.