

Question 7

Compare the following forms on $\mathbb{R}^2$:

(A) $x \, dx + y \, dy = 0$,

(B) $e^x x \, dx + e^x y \, dy = 0$,

(C) $(1 + x^2 + y^2)x \, dx + (1 + x^2 + y^2)y \, dy = 0$.

Tasks

1. Determine which forms are exact and give a potential for each exact form.
2. Prove that all three nevertheless have the same differentiable solution graphs.
3. Find their common solution graph with $y(0) = 2$ and its maximal open interval containing 0.
4. Let $dF = M \, dx + N \, dy$. Explain why multiplication by a continuous function $h(F)$ produces another exact form, and state an additional condition that guarantees equivalence of the graph equations.

Question 7 – Solution

Strategy. Separate two different questions: whether the displayed coefficients are partial derivatives of a potential, and whether multiplying the equation changes its solutions.

Step 1: Test each form. For (A), $M_y = N_x = 0$, with potential

$$F = \frac{x^2 + y^2}{2}.$$

For (B), $M_y = 0$ and $N_x = e^x y$, which are not equal on the plane, so (B) is not exact there. For (C), both cross partials are $2xy$, and a potential is

$$G = \frac{x^2 + y^2}{2} + \frac{(x^2 + y^2)^2}{4}.$$

Differentiating G gives the two coefficients in (C).

Step 2: Establish equivalence. The equations for a graph are respectively

$$x + yy' = 0, \quad e^x(x + yy') = 0, \quad (1 + x^2 + y^2)(x + yy') = 0.$$

Both multiplying factors are strictly positive at every real point. Each equation therefore holds exactly when the first does. Exactness of a particular form is not required for it to share the same solutions as an exact form.

Step 3: Select the common IVP solution. The first potential gives $x^2 + y^2 = 4$. Continuity from $y(0) = 2$ selects

$$\boxed{y = \sqrt{4 - x^2}, \quad I = (-2, 2)}.$$

Here $y' = -x/\sqrt{4 - x^2}$ verifies all three equations. At either endpoint, $y = 0$ but $x \neq 0$, so no finite derivative can satisfy $x + yy' = 0$; a differentiable graph cannot extend through it.

Step 4: Generalize using the chain rule. Choose an antiderivative H of the continuous function h . Then

$$d(H(F)) = H'(F) dF = h(F)(M dx + N dy).$$

This gives a potential for the new form. If $h(F)$ is nonzero throughout the region considered, division is valid there and the graph equations are equivalent. Allowing a zero factor can introduce extra solutions, so equivalence cannot be concluded from exactness alone.