

Question 6

On the half-plane $x > 0$, consider

$$(2y + x) dx + x dy = 0, \quad y(1) = 2.$$

An integrating factor is a nonzero function that makes a differential form exact when multiplied into both coefficients.

Tasks

1. Show that the original form is not exact. Derive, using the product rule, the condition on an integrating factor $\mu(x)$ depending only on x .
2. Find every such nonzero factor on $x > 0$, choose one, and construct a potential for the multiplied form.
3. Solve and verify the initial-value problem on $x > 0$.
4. Now consider the original equation on the whole plane. Decide whether the selected solution can extend through $x = 0$, and identify which member of the implicit solution family does extend differentiably through that point.

Question 6 – Solution

Strategy. Derive the integrating-factor condition from exactness rather than importing a formula, and return to the original equation at a zero of the chosen factor.

Step 1: Derive the factor. Originally $M_y = 2$ while $N_x = 1$. For $\mu = \mu(x)$, exactness requires

$$(\mu M)_y = 2\mu = (\mu N)_x = x\mu' + \mu.$$

Thus $x\mu' = \mu$. On $x > 0$, its nonzero solutions are $\mu = Kx$, $K \neq 0$. Choose $\boxed{\mu = x}$, which is nonzero throughout the stated half-plane.

Step 2: Integrate the exact form. The multiplied coefficients are $2xy + x^2$ and x^2 . Integrating in y gives

$$F = x^2y + g(x), \quad F_x = 2xy + g'(x) = 2xy + x^2.$$

Hence $g = x^3/3$ is a choice, giving the family $\boxed{x^2y + x^3/3 = C}$.

Step 3: Apply and verify the data. At $(1, 2)$, $C = 7/3$, so

$$\boxed{y = \frac{7}{3x^2} - \frac{x}{3}, \quad x > 0}.$$

Its derivative is $-14/(3x^3) - 1/3$, and substitution gives

$$2y + x + xy' = \frac{14}{3x^2} - \frac{2x}{3} + x - \frac{14}{3x^2} - \frac{x}{3} = 0.$$

The initial value is also 2.

Step 4: Check extension in the original equation. The selected solution tends to $+\infty$ as $x \downarrow 0$, so it cannot extend continuously through 0. In the general family, $y = C/x^2 - x/3$, a finite extension requires $C = 0$. This member extends as $\boxed{y = -x/3}$ on all of $\mathbb{R}$ and satisfies the original equation even at 0, where the equation requires $2y(0) = 0$. The multiplied form alone gives $0 = 0$ there because its factor vanishes; that weakened condition must not replace the original one.